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COMPUTATIONAL FORMULATION FOR PERIODIC VIBRATION OF GEOMETRICALLY NONLINEAR STRUCTURES—PART 1: THEORETICAL BACKGROUND

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Abstract—A general computational formulation for geometrically nonlinear structures excited by harmonic forces and executing periodic motion in a steady-state is presented. The equations of both continuous and discretized models are reformulated to obtain the motion equation in a more suitable form to a further analysis. The multi-harmonic solution of motion equation is written in a form of truncated Fourier series. Next, the Galerkin, Ritz and the harmonic balance method are discussed in a context of their equivalency in derivation of the matrix amplitude equation. The matrix amplitude equation as well as the associated tangent matrix are given in an explicit form. The stability of steady-state solution is discussed by using the Floquet theory. The numerical algorithm and an example application are described in a companion paper by Lewandowski [Lewandowski, R. Computational formulation for periodic vibration of geometrically nonlinear structures—Part 2: Numerical strategy and numerical examples. *International Journal of Solids and Structures*. (in preparation)]. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

When a nonlinear structure is subjected to harmonically varying forces, it first passes through a transient state and afterwards it reaches a steady-state or executes chaotic motions. Some steady-states can be non-periodic but mostly the periodic steady-states are observed. Two related problems of interest are the periodic behavior of structure undergoing harmonic excitation and a free vibration of structure.

Natural vibration of the undamped, nonlinear systems is of primary concern in studying the resonance phenomena because the backbone curves (the amplitude-frequency relations) and the modes of vibrations, i.e. the dynamic characteristics of systems, are determined. Analytical expressions for the backbone curves are available only for very simple systems and therefore numerical methods are necessary when considering more complex cases.

In some cases, such as the one in a range where the internal resonance exists, the corresponding backbone curves have a very complex shape owing to the presence of sharp peaks, looping characteristic and rapidly changing slopes. It is difficult to determine these types of backbone curves by previously developed methods suggested, for example, by Mei (1972) and Wellford *et al.* (1980). Up to now the considered problem can be successfully solved only by the continuation method. One version of this method is described in a paper by Lewandowski (1992).

The weakly nonlinear and periodically excited structures can be successfully analyzed by the perturbation method as proposed by Padovan (1980). The Ritz, Galerkin and the harmonic balance methods are used as the ones to be able to correctly predict the dynamic behavior of structures for which the nonlinear effects are large, as pointed out by Ling and Wu (1987), Cheung and Lau (1982) and by Lewandowski (1987).

The aim of this paper is to describe a theoretical background of systematic computer method for analyzing the free and steady-state periodic vibrations of the geometrically nonlinear structures. In particular, the higher-order solutions with many harmonics are considered. In Section 2 the equations describing both the continuous and discretized models of structure are reformulated to obtain a form of motion equations more suitable to a further analysis. In Section 3 some well known approximate analytical methods are discussed in a context of their equivalency. Moreover, the matrix amplitude equation is derived in an explicit form. Next, in Section 4 the stability analysis of steady-state solution is examined using the Floquet theory. Section 5 is devoted to derivation of the tangent matrix associated with the matrix amplitude equation.

In comparison with the existing literature this work contains a few new results. The presented general formulation relates to a wide class of structures i.e. the geometrically nonlinear ones. The general formulation given by Padovan (1980) is restricted to the structures exhibiting the weakly nonlinear behaviors in steady-states. In other formulations only the special types of structures, mainly the plates and beams, are considered. Moreover, it is proved that using the Galerkin, Ritz and the harmonic balance methods the solutions with identical accuracy are obtained when the same harmonics are taken into account in the assumed forms of steady-state solutions. Additionally, the explicit form of matrix amplitude equation and the associated tangent matrix are derived. These are very important results of presented work because many tedious algebraic operations are required in a course of derivation of the matrix amplitude equaiton and the tangent matrix. The computational formulation of stability analysis is also presented and some possible simplifications of this analysis are discussed.

2. EQUATIONS OF MOTION

2.1. *Continuous model*

Consider the motion of an elastic body in the Cartesian coordinate system. Using the total Lagrangian description we can write the following virtual work equation [see Bathe *et al.* (1975)]

$$\int_V (m\ddot{u}_k\,\delta u_k - f_k\,\delta u_k + \sigma_{ij}\,\delta e_{ij})\,\mathrm{d}V = \int_A p_k\,\delta u_k\,\mathrm{d}A, \tag{1}$$

for a body in the configuration at time t_1 but referred to the configuration at time t_0. In eqn (1) the summation convention is adopted for repeated indices. The mass density per unit volume of the body, the volume of the body, the surface area, the components of displacements, the static surface and volume forces are denoted by m, V, A, u_k, p_k, f_k, $k = 1$, 2, 3, respectively. Moreover, the Cartesian components of second Pioli–Kirchhoff stress tensor and the Green–Lagrange strains tensor are introduced and denoted by σ_{ij} and e_{ij}, respectively. All of above-mentioned quantities are measured in configuration at time t_1 and referred to the configuration at time t_0. At time t_0 the body is in undeformed state. The symbol δu_k denotes the (virtual) variation in the current displacement components u_k and δe_{ij} are the corresponding (virtual) variations in strains. A dot denotes differentiation with respect to time.

In this work we consider the geometrically nonlinear body undergoing the large displacements associated with the small strains and small rotations. The body material has perfectly elastic properties. Taking into account the above assumptions we can write the following relations:

$$\sigma_{ij} = E_{ijrs}e_{rs}, \tag{2}$$

$$e_{ij} = \varepsilon_{ij} + \eta_{ij}, \tag{3}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{4}$$

$$\eta_{ij} = \tfrac{1}{2}u_{k,i}u_{k,i}, \tag{5}$$

$$\delta e_{ij} = \delta\varepsilon_{ij} + \delta\eta_{ij}, \tag{6}$$

$$\delta\varepsilon_{ij} = \tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i}), \tag{7}$$

$$\delta\eta_{ij} = \tfrac{1}{2}(u_{k,i}\,\delta u_{k,j} + \delta u_{k,i} u_{k,j}), \tag{8}$$

where E_{ijrs} denotes the component of tensor of linear elastic material properties and $(\cdot)_{,k} = \partial(\cdot)/\partial x_k$.

The work of internal forces can be described by displacements in the following way

$$\int_V \sigma_{ij}\,\delta e_{ij}\,\mathrm{d}V = \int_V E_{ijrs}\varepsilon_{rs}\,\delta\varepsilon_{ij}\,\mathrm{d}V + \int_V E_{ijrs}(\eta_{rs}\,\delta\varepsilon_{ij} + \varepsilon_{rs}\,\delta\eta_{ij})\,\mathrm{d}V + \int_V E_{ijrs}\eta_{rs}\,\delta\eta_{ij}\,\mathrm{d}V. \tag{9}$$

The successive terms in the right-hand-side of relation (9) are the homogeneous, linear, quadratic and cubic functions of displacements u_i, respectively. The mathematical properties of these terms are important and very useful in a further analysis.

Next, we also take into account the equilibrium conditions of a body in the configuration at time $t_2 = t_1 + \Delta t$, which is assumed to be close to the configuration at time t_1. In this case the virtual work equation takes the form

$$\int_V (m\ddot{\tilde{u}}_k\,\delta\tilde{u}_k - \tilde{f}_k\,\delta\tilde{u}_k + \tilde{\sigma}_{ij}\,\delta\tilde{e}_{ij})\,\mathrm{d}V = \int_A \tilde{p}_k\,\delta\tilde{u}_k\,\mathrm{d}A, \tag{10}$$

where all quantities with a wave are measured in the current configuration at time t_2 and referred to the undeformed state of body.

A current state of body can be described incrementally as follows:

$$\tilde{\sigma}_{ij} = \sigma_{ij} + \Delta\sigma_{ij}, \tag{11}$$

$$\tilde{u}_i = u_i + \Delta u_i, \tag{12}$$

$$\tilde{e}_{ij} = e_{ij} + \Delta e_{ij}, \tag{13}$$

$$\Delta e_{ij} = \Delta\varepsilon_{ij} + \Delta\eta_{ij}, \tag{14}$$

$$\Delta\varepsilon_{ij} = \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + u_{k,i}\,\Delta u_{k,j} + \Delta u_{k,i} u_{k,j}), \tag{15}$$

$$\Delta\eta_{ij} = \tfrac{1}{2}\Delta u_{k,i}\,\Delta u_{k,j}, \tag{16}$$

$$\delta\tilde{e}_{ij} = \delta\Delta e_{ij} = \delta\Delta\varepsilon_{ij} + \delta\Delta\eta_{ij}, \tag{17}$$

$$\delta\Delta\eta_{ij} = \tfrac{1}{2}(\Delta u_{k,j}\,\delta\Delta u_{k,i} + \Delta u_{k,i}\,\delta\Delta u_{k,j}), \tag{18}$$

$$\Delta\sigma_{ij} = E_{ijrs}\,\Delta e_{rs}. \tag{19}$$

Inserting relations (11)–(19) into eqn (10) we obtain an incremental form of virtual work equation which is nonlinear with respect to the increments of displacement Δu_i as was pointed out by Bathe *et al.* (1975). After neglecting all nonlinear terms the resulting linear incremental virtual work equation can be written as

$$\int_V (m\,\Delta\ddot{u}_k\,\delta\Delta u_k + E_{ijrs}\,\Delta e_{rs}\,\delta\Delta e_{ij} + \sigma_{ij}\,\delta\Delta\eta_{ij})\,\mathrm{d}V$$

$$= \int_A \tilde{p}_k\,\delta\Delta u_k\,\mathrm{d}A - \int_V (m\ddot{u}_k\,\delta\Delta u_k - f_k\,\delta\Delta u_k + \sigma_{ij}\,\delta\Delta\varepsilon_{ij})\,\mathrm{d}V. \tag{20}$$

The second and third term on the left-hand-side of eqn (20) can be written as a sum of a few terms which are also the homogeneous, linear, quadratic and cubic functions of displacements and their increments, respectively. The resulting relations could be written as

$$\int_V E_{ijrs}\,\Delta\varepsilon_{rs}\,\delta\Delta\varepsilon_{ij}\,\mathrm{d}V = \frac{1}{2}\int_V E_{ijrs}(\Delta u_{r,s}+\Delta u_{s,r})\,\delta\Delta\varepsilon_{ij}\,\mathrm{d}V + \frac{1}{2}\int_V E_{ijrs}(u_{k,r}\,\Delta u_{k,s}+u_{k,s}\,\Delta u_{k,r})\,\delta\Delta\varepsilon_{ij}\,\mathrm{d}V, \quad (21)$$

$$\int_V \sigma_{ij}\,\delta\Delta\eta_{ij}\,\mathrm{d}V = \frac{1}{2}\int_V E_{ijrs}\varepsilon_{rs}(\Delta u_{k,j}\,\delta\Delta u_{k,i}+\Delta u_{k,i}\,\delta\Delta u_{k,j})\,\mathrm{d}V + \frac{1}{2}\int_V E_{ijrs}\eta_{rs}(\Delta u_{k,j}\,\delta\Delta u_{k,i}+\Delta u_{k,i}\,\delta\Delta u_{k,j})\,\mathrm{d}V. \quad (22)$$

In a stability analysis of motion we compare two kinds of motion of the body at the same time t. The first one is the motion in which stability is examined while the second one is the perturbed motion obtained by introducing the small perturbation into the motion of first kind. In such a case eqn (10) is a weak form of equilibrium equation in the perturbed state and eqns (11)–(19) are the relations between the perturbed and reference motions. Moreover, $\tilde{p}_k = p_k$ that, together with eqn (1), allow us to write an incremental version of perturbed motion in the form

$$\int_V (m\,\Delta\ddot{u}_k\,\delta\Delta u_k + E_{ijrs}\,\Delta e_{rs}\,\delta\Delta e_{ij} + \sigma_{ij}\,\delta\Delta\eta_{ij})\,\mathrm{d}V = 0. \quad (23)$$

2.2. *Discrete model*

Following the usual finite element procedure, the displacements u_i and their increments Δu_i within the element are given in terms of the nodal displacements $\mathbf{u}_e(t)$ and $\Delta\mathbf{u}_e(t)$, respectively, as

$$\mathbf{u}(\mathbf{x},t) = \mathbf{N}(\mathbf{x})\mathbf{u}_e(t),$$
$$\Delta\mathbf{u}(\mathbf{x},t) = \mathbf{N}(\mathbf{x})\,\Delta\mathbf{u}_e(t), \quad (24)$$

where $\mathbf{u}(\mathbf{x},t)$ and $\Delta\mathbf{u}(\mathbf{x},t)$ are the vectors of functions of displacements and their increments, $\mathbf{N}(\mathbf{x})$ is the matrix of shape functions and $\mathbf{x}$ denotes the vector of independent variables. The dimensions of vectors $\mathbf{u}$, $\Delta\mathbf{u}$, $\mathbf{x}$ and matrix $\mathbf{N}(\mathbf{x})$ depend upon the considered particular case.

In a matrix notation and by using, in general case, the following definitions:

$$\mathbf{e} = \mathrm{col}(e_{11}, e_{22}, e_{33}, 2e_{12}, 2e_{13}, 2e_{23}),$$
$$\mathbf{e}_l = \mathrm{col}(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23}),$$
$$\mathbf{e}_n = \mathrm{col}(\eta_{11}, \eta_{22}, \eta_{33}, 2\eta_{12}, 2\eta_{13}, 2\eta_{23}),$$
$$\mathbf{T} = \mathrm{col}(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}),$$

we can rewrite relations (2)–(8) and (14)–(16) as

$$\mathbf{T} = \mathbf{T}_l + \mathbf{T}_n = \mathbf{E}\mathbf{e} \quad (25)$$

$$\mathbf{e} = \mathbf{e}_l + \mathbf{e}_n, \quad (26)$$

$$\mathbf{e}_l = \mathbf{B}_0\mathbf{u}_e, \tag{27}$$

$$\mathbf{e}_n = \tfrac{1}{2}\mathbf{B}_l(\mathbf{u})\mathbf{u}_e, \tag{28}$$

$$\delta\mathbf{e} = \delta\mathbf{e}_l + \delta\mathbf{e}_n = \mathbf{B}(\mathbf{u})\,\delta\mathbf{u}_e, \tag{29}$$

$$\delta\mathbf{e}_l = \mathbf{B}_0\,\delta\mathbf{u}_e, \tag{30}$$

$$\delta\mathbf{e}_n = \mathbf{B}_1(\mathbf{u})\,\delta\mathbf{u}_e, \tag{31}$$

$$\Delta\mathbf{e} = \Delta\mathbf{e}_l + \Delta\mathbf{e}_n, \tag{32}$$

$$\Delta\mathbf{e}_l = \mathbf{B}_0\,\Delta\mathbf{u}_e + \mathbf{B}_1(\mathbf{u})\,\Delta\mathbf{u}_e, \tag{33}$$

$$\Delta\mathbf{e}_n = \tfrac{1}{2}\mathbf{B}_1(\Delta\mathbf{u})\,\Delta\mathbf{u}_e, \tag{34}$$

where $\mathbf{E}$ is the matrix of elastic material properties and $\mathbf{B}_0$, $\mathbf{B}_1(\mathbf{u})$ are the linear and nonlinear strain–displacement transformation matrices, respectively. It is well known [compare a book by Zienkiewicz and Taylor (1980)] that the matrix $\mathbf{B}_1(\mathbf{u}) = \mathbf{A}(\mathbf{u})\mathbf{G}_e$ and it is a linear and homogeneous function of nodal parameters. The form of matrices $\mathbf{A}(\mathbf{u})$ and $\mathbf{G}_e$ also depends upon a considered particular case.

The matrix description of eqns (1), (20) and (23) is

$$\mathbf{R}(\mathbf{u}) = \mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + [\mathbf{K}_0 + \mathbf{K}_1(\mathbf{u}) + \tfrac{1}{2}\mathbf{K}_2(\mathbf{u})]\mathbf{u} - \mathbf{P}(t) = \mathbf{0}, \tag{35}$$

$$\mathbf{M}\,\Delta\ddot{\mathbf{u}} + \mathbf{C}\,\Delta\dot{\mathbf{u}} + [\mathbf{K}_0 + \mathbf{K}_2(\mathbf{u}) + \mathbf{K}_3(\mathbf{u}) + \mathbf{K}_4(\mathbf{u}) + \mathbf{K}_5(\mathbf{u})]\,\Delta\mathbf{u} = \tilde{\mathbf{P}} - \mathbf{R}(\mathbf{u}), \tag{36}$$

$$\mathbf{M}\,\Delta\ddot{\mathbf{u}} + \mathbf{C}\,\Delta\dot{\mathbf{u}} + [\mathbf{K}_0 + \mathbf{K}_2(\mathbf{u}) + \mathbf{K}_3(\mathbf{u}) + \mathbf{K}_4(\mathbf{u}) + \mathbf{K}_5(\mathbf{u})]\,\Delta\mathbf{u} = \mathbf{0}, \tag{37}$$

where additionally, the viscous damping forces are introduced and $\mathbf{u}$ is the global vector of nodal parameters. The residual force vector $\mathbf{R}(\mathbf{u})$ vanishes in an equilibrium state. The global matrices $\mathbf{M}$, $\mathbf{K}_0$, $\mathbf{K}_i(\mathbf{u})$, $i = 1, 2, \ldots, 5$ and the vector $\mathbf{P}(t)$ are built in a usual way from the matrices $\mathbf{M}^e$, $\mathbf{K}_0^e$, $\mathbf{K}_i^e(\mathbf{u}_e)$ and the vector $\mathbf{P}^e(t)$ of finite elements, respectively. The definitions of these matrices and the corresponding parts of virtual work equations in the continuous model are given below:

$$\int_V m\ddot{u}_k\,\delta u_k\,\mathrm{d}V \Rightarrow \delta\ddot{\mathbf{u}}_e^t\mathbf{M}^e\ddot{\mathbf{u}}_e = \delta\mathbf{u}_e^t \int_{V_e} m\mathbf{N}^t\mathbf{N}\,\mathrm{d}V\ddot{\mathbf{u}}_e, \tag{38}$$

$$\int_V E_{ijrs}\varepsilon_{rs}\,\delta\varepsilon_{ij}\,\mathrm{d}V \Rightarrow \delta\mathbf{u}_e^t\mathbf{K}_0^e\mathbf{u}_e = \delta\mathbf{u}_e^t \int_{V_e} \mathbf{B}_0^t\mathbf{E}\mathbf{B}_0\,\mathrm{d}V\mathbf{u}_e, \tag{39}$$

$$\int_V E_{ijrs}(\eta_{rs}\,\delta\varepsilon_{ij} + \varepsilon_{rs}\,\delta\eta_{ij})\,\mathrm{d}V \Rightarrow \delta\mathbf{u}_e^t\mathbf{K}_1^e(\mathbf{u})\mathbf{u}_e = \delta\mathbf{u}_e^t\left[\frac{1}{2}\int_{V_e}\mathbf{B}_0^t\mathbf{E}\mathbf{B}_1(\mathbf{u})\,\mathrm{d}V + \int_{V_e}\mathbf{B}_1^t(\mathbf{u})\mathbf{E}\mathbf{B}_0\,\mathrm{d}V\right]\mathbf{u}_e \tag{40}$$

$$\int_V E_{ijrs}\eta_{rs}\,\delta\eta_{ij}\,\mathrm{d}V \Rightarrow \frac{1}{2}\delta\mathbf{u}_e^t\mathbf{K}_2^e(\mathbf{u})\mathbf{u}_e = \frac{1}{2}\delta\mathbf{u}_e^t\int_{V_e}\mathbf{B}_1^t(\mathbf{u})\mathbf{E}\mathbf{B}_1(\mathbf{u})\,\mathrm{d}V\mathbf{u}_e, \tag{41}$$

$$\frac{1}{2}\int_V E_{ijrs}(\Delta u_{r,s} + \Delta u_{s,r})\,\delta\Delta\varepsilon_{ij}\,\mathrm{d}V \Rightarrow \delta\Delta\mathbf{u}_e^t[\mathbf{K}_0^e + \mathbf{K}_2^e(\mathbf{u}) + \mathbf{K}_3^e(\mathbf{u}_e)]\,\Delta\mathbf{u}_e = \delta\mathbf{u}_e^t\left[\int_{V_e}\mathbf{B}_0^t\mathbf{E}\mathbf{B}_0\,\mathrm{d}V + \int_{V_e}\mathbf{B}_0^t\mathbf{E}\mathbf{B}_1(\mathbf{u})\,\mathrm{d}V + \int_{V_e}\mathbf{B}_1^t(\mathbf{u})\mathbf{E}\mathbf{B}_0\,\mathrm{d}V\right]\Delta\mathbf{u}_e, \tag{42}$$

$$\frac{1}{2}\int_V E_{ijrs}\varepsilon_{rs}(\Delta u_{k,j}\,\delta\Delta u_{k,i}+\Delta u_{k,i}\,\delta\Delta u_{k,j})\,\mathrm{d}V$$

$$\Rightarrow \delta\Delta\mathbf{u}_e^t\mathbf{K}_4^e(\mathbf{u}_e)\,\Delta\mathbf{u}_e = \delta\Delta\mathbf{u}_e^t\int_{V_e}\mathbf{G}_e^t\mathbf{Z}_l(\mathbf{u})\mathbf{G}_e\,\mathrm{d}V\,\Delta\mathbf{u}_e, \quad (43)$$

$$\frac{1}{2}\int_V E_{ijrs}\eta_{rs}(\Delta u_{k,j}\,\delta\Delta u_{k,i}+\Delta u_{k,i}\,\delta\Delta u_{k,j})\,\mathrm{d}V \Rightarrow \delta\Delta\mathbf{u}_e^t\mathbf{K}_5^e(\mathbf{u}_e)\,\Delta\mathbf{u}_e = \delta\Delta\mathbf{u}_e^t\int_{V_e}\mathbf{G}_e^t\mathbf{Z}_n(\mathbf{u}_e)\mathbf{G}_e\,\mathrm{d}V\,\Delta\mathbf{u}_e, \quad (44)$$

$$\int_V f_k\,\delta u_k\,\mathrm{d}V+\int_A p_k\,\delta u_k\,\mathrm{d}A \Rightarrow \delta\mathbf{u}_e^t\mathbf{P}^e = \delta\mathbf{u}_e^t\left[\int_{V_e}\mathbf{N}^t\mathbf{f}\,\mathrm{d}V+\int_{A_e}\mathbf{N}^t\mathbf{p}\,\mathrm{d}A\right]. \quad (45)$$

The initial stress matrices $\mathbf{Z}_l(\mathbf{u})$ and $\mathbf{Z}_n(\mathbf{u})$ which appear in relations (43) and (44) are built in the usual way using the elements of vectors $\mathbf{T}_l$ and $\mathbf{T}_n$, respectively. The former one is a linear and homogeneous function of $\mathbf{u}$ while the latter one is a homogeneous and quadratic function of $\mathbf{u}$. In general, the damping matric $\mathbf{C}$ must be derived on a basis of damping properties of structures. Only for simplicity it is assumed that $\mathbf{C} = \kappa_1\mathbf{K}_0+\kappa_2\mathbf{M}$ in our numerical calculations, where κ_1 and κ_2 denote some constants.

In many engineering applications the external forces vary harmonically with time, so we assume herein

$$\mathbf{P}(t) = \mathbf{P}_i^c\cos z_i\lambda t+\mathbf{P}_i^s\sin z_i\lambda t, \quad (46)$$

where λ denotes the fundamental excitation frequency, z_i are the integer factors, $i = 1, 2, \dots, n$ and once again the summation convention is adopted for repeated indices.

3. MATRIX AMPLITUDE EQUATION

3.1. *General considerations*

The steady-state response of nonlinear system subjected to the harmonic forces may be harmonic, subharmonic, superharmonic, almost periodic or chaotic. The periodic responses of nonlinear system could be very rich and there are numerous of phenomena such as the jump phenomenon and the internal and secondary resonances which are not present in the linear systems. However, the periodic responses of structures are very common and in many cases they exist in a full or almost full range of excitation frequency. For this reason, a knowledge of periodic steady-states is very important. Moreover, the stability analysis gives us some information about the system parameters for which other periodic or non-periodic responses are possible.

In this work, we restrict our consideration to the analysis of periodic responses and for completeness the solutions with many harmonics are taken into account so the solution of motion equation is written in the following truncated Fourier series in time

$$\mathbf{u}(t) = \mathbf{a}_i\cos z_i\lambda t+\mathbf{b}_i\sin z_i\lambda t, \quad (47)$$

where $i = 1, 2, \dots, n$.

Notice, that some harmonics which are not really present in an excitation forces description could be necessary to a correct description of structures behavior. In this case, we formally introduce these harmonics also in the description of excitation forces but with the amplitudes equal to zero.

In a similar way, the increments of displacements are assumed to be also the periodic functions with respect to time t

$$\Delta \mathbf{u}(t) = \Delta \mathbf{a}_i \cos z_i \lambda t + \Delta \mathbf{b}_i \sin z_i \lambda t. \tag{48}$$

The integer number z_i must be distinct and must include the sufficiently higher harmonics to study the expected nonlinear responses. Recently, Leung and Fung (1989) proposed the completeness and balanceability conditions to determine the number of sufficient terms n in a correct Fourier expansion of periodic solution. A few harmonics in the steady-state solution are indispensable if, for example, we look for the higher accuracy solutions or if the internal resonances are of interest.

The unknown vectors of amplitudes $\mathbf{a}_i$, $\mathbf{b}_i$ have been determined by solving a system of nonlinear, algebraic equations called the matrix amplitude equation. This equation is derived by using the Galerkin procedure in a time domain. The mathematical basis of the Galerkin method applied to find the periodic solutions of a system of nonlinear, ordinary differential equations is given by Urabe (1965). In particular, it is proved that if an isolated periodic solution exists, then there also exists the Galerkin approximate solution which converges uniformly to the exact one. The Galerkin method states that the following conditions must be satisfied:

$$\frac{2}{T}\int_0^T \mathbf{R}(\mathbf{u}(t)) \cos z_l \lambda t \, \mathrm{d}t = \mathbf{0}, \tag{49}$$

$$\frac{2}{T}\int_0^T \mathbf{R}(\mathbf{u}(t)) \sin z_l \lambda t \, \mathrm{d}t = \mathbf{0}, \tag{50}$$

where $T = 2\Pi/\lambda$ and $l = 1, 2, \ldots, n$.

The residual vector $\mathbf{R}(\mathbf{u}(t))$ could be described as a function of time if the assumed solution (47) is introduced into equation of motion (35). The residuals exist because the assumed solution is only an approximate one. An explicit form of matrix amplitude equation will be derived later.

At this stage we would like to briefly compare the Galerkin conditions resulting in the matrix amplitude equation with ones derived by other analytical methods.

3.2. *Comparison with the Ritz method*

A good starting point for finding an approximate solution by the Ritz method is the Hamilton principle which states [see book by Szemplińska-Stupnicka (1990)]

$$\delta I = \int_{t_1}^{t_2} (\delta K + \delta \mathbf{u}^t \mathbf{Q}) \, \mathrm{d}t = 0, \tag{51}$$

if $\delta \mathbf{u}(t_1) = \delta \mathbf{u}(t_2) = \mathbf{0}$, where K denotes the kinetic energy of structure and $\mathbf{Q}$ is the vector of generalized forces (not necessary conservative). The Hamilton principle can be transformed into

$$\delta I = \int_{t_1}^{t_2} \delta \mathbf{u}^t \left[\mathbf{Q} + \frac{\partial K}{\partial \mathbf{u}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \mathbf{v}}\right)\right] \mathrm{d}t = 0, \tag{52}$$

where $\mathbf{v} = \dot{\mathbf{u}}$. The expression in square brackets is a left-hand-side of the Lagrange equation so we can write

$$\mathbf{R}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \mathbf{v}}\right) - \frac{\partial K}{\partial \mathbf{u}} - \mathbf{Q} = \mathbf{0}. \tag{53}$$

The approximate solution of motion equation has the form given by (47) and we wish to determine the vectors $\mathbf{a}_i$, $\mathbf{b}_i$ in such a way that condition (52) is satisfied in the "best way".

Taking into account that

$$\delta\mathbf{u}(t) = \delta\mathbf{a}_i \cos z_i\lambda t + \delta\mathbf{b}_i \sin z_i\lambda t, \tag{54}$$

we can rewrite (52) in the form

$$\int_{t_1}^{t_2} (\delta\mathbf{a}_i \cos z_i\lambda t + \delta\mathbf{b}_i \sin z_i\lambda t)^t \mathbf{R}(\mathbf{u}(t))\,\mathrm{d}t = \mathbf{0}, \tag{55}$$

which directly leads us to the Galerkin conditions (49) and (50) because the variations $\delta\mathbf{a}_i$ and $\delta\mathbf{b}_i$ are independent and we could take $t_1 = 0$ and $t_2 = T$.

In conclusion, both considered methods give us the identical matrix amplitude equation if the assumed forms of steady state solutions are also identical. This means that the same accuracy of results is obtained by the Ritz and Galerkin methods.

3.3. *Comparison with the harmonic balance method*

The harmonic balance method is probably the oldest analytical one in the theory of nonlinear vibration. The solution of motion equation, assumed herein also in the form of eqn (47), is introduced into eqn (35) to obtain the vectors of residuals $\mathbf{R}(\mathbf{u}(t))$. The residuals vector is assumed to be a periodic function of t with a period T and it is expanded in the Fourier series in time

$$\mathbf{R}(t) = \mathbf{R}_i^c \cos z_i\lambda t + \mathbf{R}_i^s \sin z_i\lambda t, \tag{56}$$

where now $i = 1, 2, \ldots, \infty$ and the coefficients $\mathbf{R}_i^c$ and $\mathbf{R}_i^s$ are calculated using the formulas:

$$\mathbf{R}_i^c = \frac{1}{T}\int_0^T \mathbf{R}(t)\cos z_i\lambda t\,\mathrm{d}t, \tag{57}$$

$$\mathbf{R}_i^s = \frac{1}{T}\int_0^T \mathbf{R}(t)\sin z_i\lambda t\,\mathrm{d}t. \tag{58}$$

Following the harmonic balance procedure we separately equate the coefficients of $\cos z_i\lambda t$ and $\sin z_i\lambda t$ to zero for these z_i which are present in our solution. Other harmonic components in $\mathbf{R}(\mathbf{u}(t))$ are simply ignored. It is very easy to write the conditions $\mathbf{R}_i^c = \mathbf{0}$ and $\mathbf{R}_i^s = \mathbf{0}$ in a form of relations (49) and (50), respectively. That means an equivalency of the Galerkin and harmonic balance methods as the ways of derivation of the matrix amplitude equation. Moreover, the considered method gives us the possibility to verify a correctness of assumed form of steady state solution because the coefficients of ignored harmonics are also determined. This could be done by calculating the norms of vectors $\mathbf{R}_i^c$ and $\mathbf{R}_i^s$ and verifying which harmonics have significant part in the expansion (56).

The vectors $\mathbf{R}_i^c$ and $\mathbf{R}_i^s$ could be also determined in a different way if the nonlinear terms of motion equation are polynomials with respect to $\mathbf{u}$ and $\dot{\mathbf{u}}$. In this case, it is possible to introduce the solution (47) into a nonlinear term of motion equation and next transform the resulting products of trigonometric functions into a sum of these functions. For example, doing these for the term $\mathbf{K}_1(\mathbf{u})\mathbf{u}$ we have

$$\mathbf{K}_1(\mathbf{u})\mathbf{u} = [\mathbf{K}_1(\mathbf{a}_i)\mathbf{a}_j \cos z_i\lambda t \cos z_j\lambda t + \mathbf{K}_1(\mathbf{a}_i)\mathbf{b}_j \cos z_i\lambda t \sin z_j\lambda t$$
$$+\mathbf{K}_1(\mathbf{b}_i)\mathbf{a}_j \sin z_i\lambda t \cos z_j\lambda t + \mathbf{K}_1(\mathbf{a}_i)\mathbf{b}_j \sin z_i\lambda t \sin z_j\lambda t]$$
$$= \tfrac{1}{2}\{[\mathbf{K}_1(\mathbf{a}_i)\mathbf{a}_j - \mathbf{K}_1(\mathbf{b}_i)\mathbf{b}_j]\cos(z_i+z_j)\lambda t$$
$$+[\mathbf{K}_1(\mathbf{a}_i)\mathbf{a}_j + \mathbf{K}_1(\mathbf{b}_i)\mathbf{b}_j]\cos(z_i-z_j)\lambda t$$
$$+[\mathbf{K}_1(\mathbf{a}_i)\mathbf{b}_j + \mathbf{K}_1(\mathbf{b}_i)\mathbf{a}_j]\sin(z_i+z_j)\lambda t$$
$$+[-\mathbf{K}_1(\mathbf{a}_i)\mathbf{b}_j + \mathbf{K}_1(\mathbf{b}_i)\mathbf{a}_j]\sin(z_i-z_j)\lambda t\}. \qquad (59)$$

Notice, that the notation $\mathbf{K}_1(\mathbf{a}_i)$ means that an argument of matrix $\mathbf{K}_1$ is now the amplitude vector of harmonic i.

If for some i and $jz_l = |z_i \pm z_j|$ the matrix factors standing by $\cos z_l\lambda t$ and $\sin z_l\lambda t$ are the appropriate parts of vectors $\mathbf{R}_l^c$ and $\mathbf{R}_l^s$, respectively.

A detailed description of the considered version of harmonic balance method for the geometrically nonlinear structures is given in work by Lewandowski (1992).

Lau and Chung (1981) proposed a new version of harmonic balance method called the incremental harmonic balance method. A proof of equivalence of this method and the harmonic balance method is given in work by Ferri (1986).

In conclusion, the Galerkin, Ritz and the harmonic balance method are equivalent because for the identical forms of assumed steady state solution they give us the matrix amplitude equation in an identical form and consequently the solutions with equal accuracy.

3.4. *Explicit form of matrix amplitude equation*

The Galerkin method is used in this section to derive an explicit form of matrix amplitude equation.

Inserting a linear term of eqn (35) i.e. $\mathbf{R}_{lin} = \mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}_0\mathbf{u} - \mathbf{P}(t)$ into the conditions (49) and (50) we obtain

$$\frac{2}{T}\int_0^T \mathbf{R}_{lin}(t)\cos z_l\lambda t\,dt = \alpha_{il}(-z_i^2\lambda^2\mathbf{M}\mathbf{a}_i + z_i\lambda\mathbf{C}\mathbf{b}_i + \mathbf{K}_0\mathbf{a}_i - \mathbf{P}_i^c),$$

$$\frac{2}{T}\int_0^T \mathbf{R}_{lin}(t)\sin z_l\lambda t\,dt = \beta_{il}(-z_i^2\lambda^2\mathbf{M}\mathbf{b}_i - z_i\lambda\mathbf{C}\mathbf{a}_i + \mathbf{K}_0\mathbf{b}_i - \mathbf{P}_i^s), \qquad (60)$$

where

$$\alpha_{il} = \frac{2}{T}\int_0^T \cos z_i\lambda t \cos z_l\lambda t\,dt,$$

$$\beta_{il} = \frac{2}{T}\int_0^T \sin z_i\lambda t \sin z_l\lambda t\,dt. \qquad (61)$$

The integrals in (61) can be easily calculated and finally we have

$$\alpha_{il} = I(z_i+z_l) + I(z_i-z_l),$$
$$\beta_{il} = -I(z_i+z_l) + I(z_i-z_l), \qquad (62)$$

where $I(z) = 0$ if $z \neq 0$ and $I(z) = 1$ if $z = 0$.

In a similar way the quadratic term $\mathbf{R}_{n1}(t) = \mathbf{K}_1(\mathbf{u})\mathbf{u}$ and the cubic one $\mathbf{R}_{n2}(t) = \frac{1}{2}\mathbf{K}_2(\mathbf{u})\mathbf{u}$ of motion equation can be integrated with respect to t that leads to the following results

$$\frac{2}{T}\int_0^T \mathbf{R}_{n1}(t)\cos z_l\lambda t\,\mathrm{d}t = \mathbf{K}_1(\mathbf{a}_i)\mathbf{a}_j\alpha_{ijl} + \mathbf{K}_1(\mathbf{b}_i)\mathbf{b}_j\beta_{ijl},$$

$$\frac{2}{T}\int_0^T \mathbf{R}_{n1}(t)\sin z_l\lambda t\,\mathrm{d}t = \mathbf{K}_1(\mathbf{a}_i)\mathbf{b}_j\gamma_{ijl} + \mathbf{K}_1(\mathbf{b}_i)\mathbf{a}_j\delta_{ijl}, \tag{63}$$

where

$$\alpha_{ijl} = \frac{2}{T}\int_0^T \cos z_i\lambda t\cos z_j\lambda t\cos z_l\lambda t\,\mathrm{d}t = \tfrac{1}{2}[I(z_i+z_j+z_l)+I(z_i+z_j-z_l)$$
$$+I(z_i-z_j+z_l)+I(z_i-z_j-z_l)],$$
$$\beta_{ijl} = \frac{2}{T}\int_0^T \sin z_i\lambda t\sin z_j\lambda t\cos z_l\lambda t\,\mathrm{d}t = \tfrac{1}{2}[-I(z_i+z_j+z_l)-I(z_i+z_j-z_l)$$
$$+I(z_i-z_j+z_l)+I(z_i-z_j-z_l)],$$
$$\gamma_{ijl} = \frac{2}{T}\int_0^T \cos z_i\lambda t\sin z_j\lambda t\sin z_l\lambda t\,\mathrm{d}t = \tfrac{1}{2}[-I(z_i+z_j+z_l)+I(z_i+z_j-z_l)$$
$$+I(z_i-z_j+z_l)-I(z_i-z_j-z_l)],$$
$$\delta_{ijl} = \frac{2}{T}\int_0^T \sin z_i\lambda t\cos z_j\lambda t\sin z_l\lambda t\,\mathrm{d}t = \frac{1}{2}[-I(z_i+z_j+z_l)+I(z_i+z_j-z_l)$$
$$-I(z_i-z_j+z_l)+I(z_i-z_j-z_l)], \tag{64}$$

and

$$\frac{2}{T}\int_0^T \mathbf{R}_{n2}(t)\cos z_l\lambda t\,\mathrm{d}t = \tfrac{1}{2}[\mathbf{K}_2(\mathbf{a}_i,\mathbf{a}_j)\mathbf{a}_k\alpha_{ijkl} + \mathbf{K}_2(\mathbf{a}_i,\mathbf{b}_j)\mathbf{b}_k\beta_{ijkl}$$
$$+\mathbf{K}_2(\mathbf{b}_i,\mathbf{a}_j)\mathbf{b}_k\gamma_{ijkl} + \mathbf{K}_2(\mathbf{b}_i,\mathbf{b}_j)\mathbf{a}_k\delta_{ijkl}],$$
$$\frac{2}{T}\int_0^T \mathbf{R}_{n2}(t)\sin z_l\lambda t\,\mathrm{d}t = \tfrac{1}{2}[\mathbf{K}_2(\mathbf{a}_i,\mathbf{b}_j)\mathbf{a}_k\mu_{ijkl} + \mathbf{K}_2(\mathbf{b}_i,\mathbf{a}_j)\mathbf{a}_k\nu_{ijkl}$$
$$+\mathbf{K}_2(\mathbf{a}_i,\mathbf{a}_j)\mathbf{b}_k\kappa_{ijkl} + \mathbf{K}_2(\mathbf{b}_i,\mathbf{b}_j)\mathbf{b}_k\omega_{ijkl}], \tag{65}$$

where

$$\alpha_{ijkl} = \frac{2}{T}\int_0^T \cos z_i\lambda t\cos z_j\lambda t\cos z_k\lambda t\cos z_l\lambda t\,\mathrm{d}t$$
$$= \tfrac{1}{4}(I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8),$$
$$\beta_{ijkl} = \frac{2}{T}\int_0^T \cos z_i\lambda t\sin z_j\lambda t\sin z_k\lambda t\cos z_l\lambda t\,\mathrm{d}t$$
$$= \tfrac{1}{4}(-I_1+I_2+I_3-I_4-I_5+I_6+I_7-I_8),$$
$$\gamma_{ijkl} = \frac{2}{T}\int_0^T \sin z_i\lambda t\cos z_j\lambda t\sin z_k\lambda t\cos z_l\lambda t\,\mathrm{d}t$$
$$= \tfrac{1}{4}(-I_1+I_2-I_3+I_4-I_5+I_6-I_7+I_8),$$

$$\delta_{ijkl} = \frac{2}{T}\int_0^T \sin z_i\lambda t \sin z_j\lambda t \cos z_k\lambda t \cos z_l\lambda t \, dt$$

$$= \tfrac{1}{4}(-I_1 - I_2 + I_3 + I_4 - I_5 - I_6 + I_7 + I_8),$$

$$\kappa_{ijkl} = \frac{2}{T}\int_0^T \cos z_i\lambda t \cos z_j\lambda t \sin z_k\lambda t \sin z_l\lambda t \, dt$$

$$= \tfrac{1}{4}(-I_1 - I_2 - I_3 - I_4 + I_5 + I_6 + I_7 + I_8),$$

$$\mu_{ijkl} = \frac{2}{T}\int_0^T \cos z_i\lambda t \sin z_j\lambda t \cos z_k\lambda t \sin z_l\lambda t \, dt$$

$$= \tfrac{1}{4}(-I_1 + I_2 + I_3 - I_4 + I_5 - I_6 - I_7 + I_8),$$

$$\nu_{ijkl} = \frac{2}{T}\int_0^T \sin z_i\lambda t \cos z_j\lambda t \cos z_k\lambda t \sin z_l\lambda t \, dt$$

$$= \tfrac{1}{4}(-I_1 + I_2 - I_3 + I_4 + I_5 - I_6 + I_7 - I_8),$$

$$\omega_{ijkl} = \frac{2}{T}\int_0^T \sin z_i\lambda t \sin z_j\lambda t \sin z_k\lambda t \sin z_l\lambda t \, dt$$

$$= \tfrac{1}{4}(I_1 + I_2 - I_3 - I_4 - I_5 - I_6 + I_7 + I_8). \tag{66}$$

In above relations the following notations:

$$I_1 = I(z_i + z_j + z_k + z_l), \quad I_2 = I(z_i + z_j - z_k - z_l),$$

$$I_3 = I(z_i - z_j + z_k + z_l), \quad I_4 = I(z_i - z_j - z_k - z_l),$$

$$I_5 = I(z_i + z_j + z_k - z_l), \quad I_6 = I(z_i + z_j - z_k + z_l),$$

$$I_7 = I(z_i - z_j + z_k - z_l), \quad I_8 = I(z_i - z_j - z_k + z_l), \tag{67}$$

are introduced.

The matrix notation like $\mathbf{K}_2(\mathbf{a}_i, \mathbf{b}_k)$ emphasized that the considered matrix is a quadratic function of amplitudes and it is the result of assembling process of matrices of finite elements defined by

$$\mathbf{K}_2^e(\mathbf{a}_i^e, \mathbf{b}_j^e) = \int_{V_e} \mathbf{B}_1^T(\mathbf{a}_i^e)\mathbf{E}\mathbf{B}_1(\mathbf{b}_j^e)\, dV. \tag{68}$$

The remaining integrals of type (61), (64) and (66) not shown above, but resulting from conditions (49) and (50), are equal to zero.

Taking into account all of these partial results we can write the matrix amplitude equations associated with the base functions $\cos z_l\lambda t$ and $\sin z_l\lambda t$ in the following form:

$$\mathbf{G}_l^c = (\mathbf{K}_0 - z_l^2\lambda^2\mathbf{M})\alpha_{il}\mathbf{a}_i + \alpha_{il}z_i\lambda\mathbf{C}\mathbf{b}_i + \mathbf{S}_l^c - \mathbf{P}_l^c = \mathbf{0},$$

$$\mathbf{G}_l^s = -\beta_{il}z_i\lambda\mathbf{C}\mathbf{a}_i + (\mathbf{K}_0 - z_i^2\lambda^2\mathbf{M})\beta_{il}\mathbf{b}_i + \mathbf{S}_l^s - \mathbf{P}_l^s = \mathbf{0}, \tag{69}$$

where $l = 1, 2, \ldots, n$,

$$\mathbf{S}_l^c = \mathbf{H}_{lk}^{cc}\mathbf{a}_k + \mathbf{H}_{lk}^{cs}\mathbf{b}_k,$$

$$\mathbf{S}_l^s = \mathbf{H}_{lk}^{sc}\mathbf{a}_k + \mathbf{H}_{lk}^{ss}\mathbf{b}_k, \tag{70}$$

$$\mathbf{H}_{lk}^{cc} = \mathbf{K}_1(\mathbf{a}_i)\alpha_{ikl} + \tfrac{1}{2}[\alpha_{ijkl}\mathbf{K}_2(\mathbf{a}_i, \mathbf{a}_j) + \delta_{ijkl}\mathbf{K}_2(\mathbf{b}_i, \mathbf{b}_j)],$$

$$\mathbf{H}_{lk}^{cs} = \mathbf{K}_1(\mathbf{b}_i)\beta_{ikl} + \tfrac{1}{2}[\beta_{ijkl}\mathbf{K}_2(\mathbf{a}_i, \mathbf{b}_j) + \gamma_{ijkl}\mathbf{K}_2(\mathbf{b}_i, \mathbf{a}_j)],$$

$$\mathbf{H}_{lk}^{sc} = \mathbf{K}_1(\mathbf{b}_i)\,\delta_{ikl} + \tfrac{1}{2}[\mu_{ijkl}\mathbf{K}_2(\mathbf{a}_i, \mathbf{b}_j) + \nu_{ijkl}\mathbf{K}_2(\mathbf{b}_i, \mathbf{a}_j)],$$

$$\mathbf{H}_{lk}^{ss} = \mathbf{K}_1(\mathbf{a}_i)\gamma_{ikl} + \tfrac{1}{2}[\kappa_{ijkl}\mathbf{K}_2(\mathbf{a}_i, \mathbf{a}_j) + \omega_{ijkl}\mathbf{K}_2(\mathbf{b}_i, \mathbf{b}_j)]. \quad (71)$$

The following equalities

$$\beta_{ill} = \delta_{ill}, \quad \beta_{ijll} = \mu_{ijll}, \quad \gamma_{ijll} = \nu_{ijll}, \quad (72)$$

arise from the definitions of above coefficients given by relations (64) and (66). This means that

$$\mathbf{H}_{ll}^{cs} = \mathbf{H}_{ll}^{sc}. \quad (73)$$

In a similar way we can prove that

$$\mathbf{H}_{lk}^{cc} = \mathbf{H}_{kl}^{cc}, \quad \mathbf{H}_{lk}^{ss} = \mathbf{H}_{kl}^{ss}, \quad \mathbf{H}_{lk}^{cs} = \mathbf{H}_{kl}^{sc}, \quad \mathbf{H}_{lk}^{sc} = \mathbf{H}_{kl}^{cs}. \quad (74)$$

It is necessary to briefly describe a case when a constant term is present in the assumed solution of motion equation, i.e. when for instance $z_n = 0$. The vector $\mathbf{a}_n$ is the constant term whereas the vector $\mathbf{b}_n$ has no influence on solution and can be chosen arbitrarily. For convenience, it is assumed that $\mathbf{b}_n = \mathbf{0}$. The factors of amplitude equations given by the relations (61), (64) and (66) can be calculated without difficulties. The Galerkin conditions for $l = n$ have the form:

$$\frac{2}{T}\int_0^T \mathbf{R}(t)\,\mathrm{d}t = \mathbf{0}, \quad \frac{2}{T}\int_0^T \mathbf{0}\,\mathrm{d}t = \mathbf{0} \quad (75)$$

and as a matter of fact only the first one must be fulfilled because the second condition goes to an identity. In the numerical algorithm, for $z_n = 0$ and $l = n$ the eqn (69_2) is simply omitted. Moreover, the remainder terms containing the vector $\mathbf{b}_n$ are also neglected.

The eqns (69) will now be written in a more compact form. Introducing the following notation:

$$\tilde{\mathbf{G}}_l = \mathrm{col}(\mathbf{G}_l^c, \mathbf{G}_l^s), \quad \tilde{\mathbf{P}}_l = \mathrm{col}(\mathbf{P}_l^c, \mathbf{P}_l^s), \quad \tilde{\mathbf{a}}_l = \mathrm{col}(\mathbf{a}_l, \mathbf{b}_l), \quad \tilde{\mathbf{S}}_l = \mathrm{col}(\mathbf{S}_l^c, \mathbf{S}_l^s)$$

$$\tilde{\mathbf{H}}_{lk} = \begin{bmatrix} \mathbf{H}_{lk}^{cc} + \alpha_{lk}\mathbf{K}_0 & \mathbf{H}_{lk}^{cs} \\ \mathbf{H}_{lk}^{sc} & \mathbf{H}_{lk}^{ss} + \beta_{lk}\mathbf{K}_0 \end{bmatrix}, \quad \tilde{\mathbf{B}}_{lk} = \begin{bmatrix} \alpha_{lk}\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \beta_{lk}\mathbf{M} \end{bmatrix}, \quad \tilde{\mathbf{D}}_{lk} = \begin{bmatrix} \mathbf{0} & \alpha_{lk}\mathbf{C} \\ -\beta_{lk}\mathbf{C} & \mathbf{0} \end{bmatrix}, \quad (76)$$

the amplitude equation associated with the harmonic l can be rewritten as follows

$$\tilde{\mathbf{G}}_l = (\tilde{\mathbf{H}}_{lk} + \lambda z_k \tilde{\mathbf{D}}_{lk} - \lambda^2 z_k^2 \tilde{\mathbf{B}}_{lk})\tilde{\mathbf{a}}_k - \tilde{\mathbf{P}}_l = \mathbf{0}, \quad (77)$$

where $l = 1, 2, \ldots, n$.

Notice, that for $k \neq l$, $\alpha_{kl} = \beta_{kl} = 0$ and the matrices $\tilde{\mathbf{B}}_{lk}$ and $\tilde{\mathbf{D}}_{lk}$ are not equal to zero only for $k = l$.

The most compact form of considered equations is obtained after introducing:

$$\mathbf{G} = \mathrm{col}(\tilde{\mathbf{G}}_1, \ldots, \tilde{\mathbf{G}}_n), \quad \mathbf{P} = \mathrm{col}(\tilde{\mathbf{P}}_1, \ldots, \tilde{\mathbf{P}}_n), \quad \mathbf{a} = \mathrm{col}(\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_n),$$

$$\mathbf{H}(\mathbf{a}) = \begin{bmatrix} \tilde{\mathbf{H}}_{11}, \ldots, \tilde{\mathbf{H}}_{1n} \\ \vdots \quad \vdots \\ \tilde{\mathbf{H}}_{n1}, \ldots, \tilde{\mathbf{H}}_{nn} \end{bmatrix}, \quad \begin{aligned} \mathbf{D} &= \lceil z_1 \tilde{\mathbf{D}}_{11}, \ldots, z_n \tilde{\mathbf{D}}_{nn} \rfloor, \\ \mathbf{B} &= \lceil z_1 \tilde{\mathbf{B}}_{11}, \ldots, z_n \tilde{\mathbf{B}}_{nn} \rfloor, \end{aligned} \tag{78}$$

where the symbol $\lceil \ldots \rfloor$ denotes the block-diagonal matrix. This notation enables us to write the matrix amplitude equation in its final form as

$$\mathbf{G}(\lambda, \mathbf{a}, \mathbf{P}) = [\mathbf{H}(\mathbf{a}) + \lambda \mathbf{D} - \lambda^2 \mathbf{B})\mathbf{a} - \mathbf{P} = \mathbf{0}. \tag{79}$$

The matrix **B** is symmetric whereas the matrix **D** is the anti-symmetric one. Moreover, the matrix $\mathbf{H}(\mathbf{a})$ is the block-symmetric one that follows from relations (73) and (74).

The considered problem will be solved if the solution of matrix amplitude eqn (79) has been found. A proposed formulation is general because a wide class of structures is considered and the multi harmonics solutions are taken into account. The integrals resulting from the Galerkin conditions (49) and (50) are calculated analytically and all of the appearing matrices can be determined, after relatively small modifications, by using the typical finite element procedures.

At the end of this section, the problem of undamped free vibration of structures is briefly considered. In this case $\mathbf{C} = \mathbf{0}$, $\mathbf{P}(t) = \mathbf{0}$. The solution of motion equation is assumed in the form:

$$\mathbf{u}(t) = \mathbf{a}_i \cos z_i \omega t, \tag{80}$$

and the amplitude equation associated with the harmonic l is given by

$$\mathbf{G}_l^c = (\mathbf{K}_0 - z_l^2 \omega^2 \mathbf{M})\alpha_{il}\mathbf{a}_i + \mathbf{S}_l^c = \mathbf{0}, \tag{81}$$

where $l = 1, 2, \ldots, n$ and $\mathbf{S}_l^c$ is defined by

$$\mathbf{S}_l^c = \mathbf{H}_{lk}^{cc}\mathbf{a}_k, \tag{82}$$

$$\mathbf{H}_{lk}^{cc} = \mathbf{K}_1(\mathbf{a}_i)\alpha_{ikl} + \tfrac{1}{2}\alpha_{ijkl}\mathbf{K}_2(\mathbf{a}_i, \mathbf{a}_j). \tag{83}$$

After introducing the notation

$$\mathbf{G} = \mathrm{col}(\mathbf{G}_1^c, \ldots, \mathbf{G}_n^c), \quad \mathbf{S} = \mathrm{col}(\mathbf{S}_1^c, \ldots, \mathbf{S}_n^c), \quad \mathbf{a} = \mathrm{col}(\mathbf{a}_1, \ldots, \mathbf{a}_n),$$

$$\mathbf{K} = \lceil \alpha_{11}\mathbf{K}_0, \ldots, \alpha_{nn}\mathbf{K}_0 \rfloor, \quad \mathbf{B} = \lceil z_1^2 \mathbf{M}\alpha_{11}, \ldots, z_n^2 \mathbf{M}\alpha_{nn} \rfloor, \mathbf{H}(\mathbf{a}) = \begin{bmatrix} \mathbf{H}_{11}^{cc} + \alpha_{11}\mathbf{K}_0, \ldots, \mathbf{H}_{1n}^{cc} \\ \vdots \quad \vdots \\ \mathbf{H}_{n1}^{cc}, \ldots, \mathbf{H}_{nn}^{cc} + \alpha_{nn}\mathbf{K}_0 \end{bmatrix}, \tag{84}$$

the compact form of the matrix amplitude can be written as

$$\mathbf{G}(\mathbf{a}, \lambda) = [\mathbf{H}(\mathbf{a}) - \omega^2 \mathbf{B}]\mathbf{a} = \mathbf{0}. \tag{85}$$

The eqn (85) can be treated as a nonlinear eigenvalue problem where ω and **a** are the eigenvalue and the eigenvector, respectively, the quantities ω and **a** are called the nonlinear frequency of vibration and the nonlinear mode of vibration in a sense suggested for the first time by Rosenberg (1966). Some additional particular cases of amplitude equations are also discussed by Lewandowski (1993).

4. STABILITY OF STEADY-STATE SOLUTIONS

The investigation of stability of steady-state solution is an important part of analysis because without it we cannot be sure that this solution is stable. The stability of steady-state solution is governed by the asymptotic behavior of disturbance $\Delta\mathbf{u}$ as time increases. This can be done by evaluating the transition matrix as pointed out by Friedman *et al.* (1977) or by calculating the characteristic exponents [see book by Bolotin (1964)]. For the former method the variational eqn (37) is integrated over one period with various initial conditions to get the transition matrix. The steady-state solution is stable if the modulus of all eigenvalues of transition matrix are smaller than unity. In this method a process of calculation of transition matrix is very costly and not appropriate for the systems with large numbers of unknowns. Moreover, in the work by Hamdan and Burton (1993) some incorrectness are found when this method is used to analyze the stability of steady-state solution given by an analytical expression. In the second method, the evolution of disturbance in time is written as a product of periodic and exponential components like

$$\delta\mathbf{u}(t) = e^{\mu t}\mathbf{q}(t), \tag{86}$$

where μ is the characteristic exponent to be determined. Applying the Galerkin method, an eigenvalue problem is derived. If all real parts of characteristic exponents are smaller than zero, the solution is stable.

In this section, the stability of steady-state solution of forced vibrations is examined by using the Floquet theory and the variational eqn (37) which, for convenience, is rewritten in the form

$$\mathbf{M}\,\delta\ddot{\mathbf{u}}+\mathbf{C}\,\delta\dot{\mathbf{u}}+\mathbf{K}_t(\mathbf{u})\,\delta\mathbf{u} = \mathbf{0}, \tag{87}$$

where the tangent stiffness matrix is given by

$$\mathbf{K}_t(\mathbf{u}) = \mathbf{K}_0+\mathbf{K}_2(\mathbf{u})+\mathbf{K}_3(\mathbf{u})+\mathbf{K}_4(\mathbf{u})+\mathbf{K}_5(\mathbf{u}). \tag{88}$$

Inserting eqn (86) into (87) one obtains

$$\delta\mathbf{R}(t) = \mathbf{M}\ddot{\mathbf{q}}(t)+(2\mu\mathbf{M}+\mathbf{C})\dot{\mathbf{q}}(t)+[\mu^2\mathbf{M}+\mu\mathbf{C}+\mathbf{K}_t(\mathbf{u})]\mathbf{q}(t) = \mathbf{0}. \tag{89}$$

The function $\mathbf{q}(t)$ is a periodic one and can be expanded into the following Fourier series

$$\mathbf{q}(t) = (\mathbf{q}_i^c \cos z_i\lambda t+\mathbf{q}_i^s \sin z_i\lambda t), \tag{90}$$

where now $i = 1, 2, \ldots, \infty$.

In numerical calculation it is possible to take into account only a finite number of terms. Often, only the harmonics present in the steady-state solution are taken into account in expression (90) what enables a determination of the principal instability regions. However, it is also possible to find the points on the response curves in which the Hopf bifurcation occurs. The additional harmonics must be included in the vector $\mathbf{q}(t)$ if the higher order instability regions are of interest [see book by Szemplińska-Stupnicka (1990)].

In order to preserve a generality of presented method the $m > n$ harmonics are taken into account in relation (90). Moreover, for convenience, the steady-state solution is formally supplemented by the harmonics which are not present in comparison with expansion (90). The amplitudes of these harmonics are assumed to be equal to zero i.e. $\mathbf{a}_{n+1} = \cdots = \mathbf{a}_m = \mathbf{b}_{n+1} = \cdots = \mathbf{b}_m = \mathbf{0}$.

The unknown vectors $\mathbf{q}_i^c$ and $\mathbf{q}_i^s$ ($i = 1, \ldots, m$) are determined from the algebraic matrix equation which will be derived using again the Galerkin method. In this case the Galerkin conditions have the form

$$\frac{2}{T}\int_0^T \delta\mathbf{R}(t)\cos z_l\lambda t\,\mathrm{d}t = \mathbf{0}, \tag{91}$$

$$\frac{2}{T}\int_0^T \delta\mathbf{R}(t)\sin z_l\lambda t\,\mathrm{d}t = \mathbf{0}. \tag{92}$$

After integrating a linear part of eqn (37) from conditions (91) and (92) one has

$$\begin{aligned}
&\frac{2}{T}\int_0^T [\mathbf{M}\ddot{\mathbf{q}}(t)+(2\mu\mathbf{M}+\mathbf{C})\dot{\mathbf{q}}(\mathbf{t})+(\mu^2\mathbf{M}+\mu\mathbf{C})\mathbf{q}(t)]\cos z_l\lambda t\,\mathrm{d}t\\
&\quad = \alpha_{il}\{[(\mu^2-z_i^2\lambda^2)\mathbf{M}+\mu\mathbf{C}]\mathbf{q}_i^c+z_i\lambda(2\mu\mathbf{M}+\mathbf{C})\mathbf{q}_i^s\},\\
&\frac{2}{T}\int_0^T [\mathbf{M}\ddot{\mathbf{q}}(t)+(2\mu\mathbf{M}+\mathbf{C})\dot{\mathbf{q}}(t)+(\mu^2\mathbf{M}+\mu\mathbf{C})\mathbf{q}(t)]\sin z_l\lambda t\,\mathrm{d}t\\
&\quad = \beta_{il}\{-z_i\lambda(2\mu\mathbf{M}+\mathbf{C})\mathbf{q}_i^c+[(\mu^2-z_i^2\lambda^2)\mathbf{M}+\mu\mathbf{C}]\mathbf{q}_i^s\}.
\end{aligned} \tag{93}$$

The nonlinear part of Galerkin condition (91) can be written as

$$\frac{2}{T}\int_0^T \mathbf{K}_t(\mathbf{u})\mathbf{q}(t)\cos z_l\lambda t\,\mathrm{d}t = \frac{2}{T}\int_0^T \frac{\partial\mathbf{F}}{\partial\mathbf{u}}\cos z_i\lambda t\cos z_l\lambda t\,\mathrm{d}t\mathbf{q}_i^c + \frac{2}{T}\int_0^T \frac{\partial\mathbf{F}}{\partial\mathbf{u}}\cos z_l\lambda t\sin z_i\lambda t\,\mathrm{d}t\mathbf{q}_i^s, \tag{94}$$

where

$$\mathbf{F}(\mathbf{u}) = [\mathbf{K}_0+\mathbf{K}_1(\mathbf{u})+\tfrac{1}{2}\mathbf{K}_2(\mathbf{u})]\mathbf{u}, \tag{95}$$

denotes the global vector of internal nodal forces. Taking into account that

$$\frac{\partial\mathbf{u}}{\partial\mathbf{a}_i} = \mathbf{I}\cos z_i\lambda t,\quad \frac{\partial\mathbf{u}}{\partial\mathbf{b}_i} = \mathbf{I}\sin z_i\lambda t,\quad \mathbf{I} = \lceil 1,\ldots,1\rfloor, \tag{96}$$

we can rewrite the right-hand-side of relation (94) in the form

$$\frac{\partial}{\partial\mathbf{a}_i}\left\{\frac{2}{T}\int_0^T \mathbf{F}(\mathbf{u}(\mathbf{a}_i,\mathbf{b}_i))\cos z_l\lambda t\,\mathrm{d}t\right\}\mathbf{q}_i^c + \frac{\partial}{\partial\mathbf{b}_i}\left\{\frac{2}{T}\int_0^T \mathbf{F}(\mathbf{u}(\mathbf{a}_i,\mathbf{b}_i))\cos z_l\lambda t\,\mathrm{d}t\right\}\mathbf{q}_i^s. \tag{97}$$

In brackets are the integrals which appear in the Galerkin condition (49). Using the relations given in Section 3.4 one can write

$$\mathbf{F}_l^c = \frac{2}{T}\int_0^T \mathbf{F}(\mathbf{u}(\mathbf{a}_i,\mathbf{b}_i))\cos z_l\lambda t\,\mathrm{d}t = \alpha_{il}\mathbf{K}_0\mathbf{a}_i+\mathbf{S}_l^c, \tag{98}$$

where $\mathbf{S}_l^c$ is defined by relation (70_1).

After introducing the following notation

$$\mathbf{K}_{li}^{cc} = \frac{\partial}{\partial\mathbf{a}_i}(\mathbf{F}_l^c),\quad \mathbf{K}_{li}^{cs} = \frac{\partial}{\partial\mathbf{b}_i}(\mathbf{F}_l^c), \tag{99}$$

the final form of eqn (91) is given by

$$\{\mathbf{K}_{li}^{cc}+\alpha_{il}[(\mu^2-z_i^2\lambda^2)\mathbf{M}+\mu\mathbf{C}]\}\mathbf{q}_i^c+\{\mathbf{K}_{li}^{cs}+z_i\lambda\alpha_{li}(2\mu\mathbf{M}+\mathbf{C})\}\mathbf{q}_i^s=\mathbf{0}. \tag{100}$$

Proceeding similarly, from the condition (92) yields

$$\frac{2}{T}\int_0^T \mathbf{K}_t(\mathbf{u})\mathbf{q}(t)\sin z_l\lambda t\,\mathrm{d}t=\mathbf{K}_{li}^{sc}\mathbf{q}_i^c+\mathbf{K}_{li}^{ss}\mathbf{q}_i^s \tag{101}$$

and finally

$$\{\mathbf{K}_{li}^{sc}-\beta_{li}z_i\lambda(2\mu\mathbf{M}+\mathbf{C})\}\mathbf{q}_i^c+\{\mathbf{K}_{li}^{ss}+\beta_{li}[(\mu^2-z_i^2\lambda^2)\mathbf{M}+\mu\mathbf{C}]\}\mathbf{q}_i^s=\mathbf{0} \tag{102}$$

where

$$\mathbf{K}_{li}^{sc}=\frac{\partial}{\partial\mathbf{a}_i}(\mathbf{F}_l^s),\quad \mathbf{K}_{li}^{ss}=\frac{\partial}{\partial\mathbf{b}_i}(\mathbf{F}_l^s),\quad l=1,\ldots,m. \tag{103}$$

The most compact form of algebraic equation resulting from the Galerkin conditions (91) and (92) is as follows

$$[\mu^2\mathbf{B}+\mu(\mathbf{Z}+2\lambda\mathbf{E})+\mathbf{K}-\lambda^2\mathbf{X}+\lambda\mathbf{D}]\mathbf{q}=\mathbf{0}. \tag{104}$$

where

$$\begin{gathered}\mathbf{q}=\mathrm{col}(\tilde{\mathbf{q}}_1,\ldots,\tilde{\mathbf{q}}_m),\quad \mathbf{B}=\lceil\tilde{\mathbf{B}}_{11},\ldots,\tilde{\mathbf{B}}_{mm}\rfloor,\\ \mathbf{Z}=\lceil\tilde{\mathbf{Z}}_{11},\ldots,\tilde{\mathbf{Z}}_{mm}\rfloor,\quad \mathbf{E}=\lceil z_1\tilde{\mathbf{E}}_{11},\ldots,z_m\tilde{\mathbf{E}}_{mm}\rfloor,\\ \mathbf{D}=\lceil z_1\tilde{\mathbf{D}}_{11},\ldots,z_m\tilde{\mathbf{D}}_{mm}\rfloor,\quad \mathbf{X}=\lceil z_1^2\tilde{\mathbf{B}}_{11},\ldots,z_m^2\tilde{\mathbf{B}}_{mm}\rfloor,\\ \mathbf{K}=\begin{bmatrix}\tilde{\mathbf{K}}_{11},\ldots,\tilde{\mathbf{K}}_{1m}\\ \vdots \qquad \vdots\\ \tilde{\mathbf{K}}_{m1},\ldots,\tilde{\mathbf{K}}_{mm}\end{bmatrix},\end{gathered} \tag{105}$$

and the symbol $\lceil\ldots\rfloor$, denotes the block-diagonal matrix. Moreover, the following additional notations are introduced

$$\tilde{\mathbf{q}}_i=\mathrm{col}(\mathbf{q}_i^c,\mathbf{q}_i^s),\quad \tilde{\mathbf{B}}_{li}=\begin{bmatrix}\alpha_{li}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \beta_{li}\mathbf{M}\end{bmatrix},\quad \tilde{\mathbf{D}}_{li}=\begin{bmatrix}\mathbf{0} & \alpha_{li}\mathbf{C}\\ -\beta_{li}\mathbf{C} & \mathbf{0}\end{bmatrix},$$

$$\tilde{\mathbf{Z}}_{li}=\begin{bmatrix}\alpha_{li}\mathbf{C} & \mathbf{0}\\ \mathbf{0} & \beta_{li}\mathbf{C}\end{bmatrix},\quad \tilde{\mathbf{E}}_{li}=\begin{bmatrix}\mathbf{0} & \alpha_{li}\mathbf{M}\\ -\beta_{li}\mathbf{M} & \mathbf{0}\end{bmatrix},\quad \tilde{\mathbf{K}}_{li}=\begin{bmatrix}\mathbf{K}_{li}^{cc} & \mathbf{K}_{li}^{cs}\\ \mathbf{K}_{li}^{sc} & \mathbf{K}_{li}^{ss}\end{bmatrix}. \tag{106}$$

In the resulting quadratic eigenvalue problem (104) the matrices **B**, **Z** and **X** are symmetric because their sub-matrices **M** and **C** are also symmetric. Moreover, it is easy to check that the matrices **E** and **D** are skew-symmetric. However, the matrix **K** is also symmetric. It can be proved by studying again the right-hand-side of eqn (94). In the relation (47) the index i can be substituted by l and then

$$\frac{\partial\mathbf{u}}{\partial\mathbf{a}_l}=\mathbf{I}\cos z_l\lambda t,\quad \frac{\partial\mathbf{u}}{\partial\mathbf{b}_l}=\mathbf{I}\sin z_l\lambda t. \tag{107}$$

The right-hand-side of eqn (94) can be rearranged into the form

$$\frac{\partial}{\partial \mathbf{a}_l}\left\{\frac{2}{T}\int_0^T \mathbf{F}(\mathbf{u}(\mathbf{a}_l,\mathbf{b}_l))\cos z_i\lambda t\,\mathrm{d}t\right\}\mathbf{q}_i^{\mathrm{c}}+\frac{\partial}{\partial \mathbf{a}_l}\left\{\frac{2}{T}\int_0^T \mathbf{F}(\mathbf{u}(\mathbf{a}_l,\mathbf{b}_l))\sin z_i\lambda t\,\mathrm{d}t\right\}\mathbf{q}_i^{\mathrm{s}}$$

$$=\frac{\partial}{\partial \mathbf{a}_l}(\mathbf{F}_i^{\mathrm{c}})\mathbf{q}_i^{\mathrm{c}}+\frac{\partial}{\partial \mathbf{a}_l}(\mathbf{F}_i^{\mathrm{s}})\mathbf{q}_i^{\mathrm{s}}=\mathbf{K}_{il}^{\mathrm{cc}}\mathbf{q}_i^{\mathrm{c}}+\mathbf{K}_{il}^{\mathrm{sc}}\mathbf{q}_i^{\mathrm{s}}. \tag{108}$$

Previously, the right-hand-side of eqn (94) was written as $\mathbf{K}_{li}^{\mathrm{cc}}\mathbf{q}_i^{\mathrm{c}}+\mathbf{K}_{li}^{\mathrm{cs}}\mathbf{q}_i^{\mathrm{s}}$. Both descriptions will be identical if and only if $\mathbf{K}_{il}^{\mathrm{cc}}=\mathbf{K}_{li}^{\mathrm{cc}}$ and $\mathbf{K}_{il}^{\mathrm{sc}}=\mathbf{K}_{li}^{\mathrm{sc}}$.

Analogously, the left-hand-side of eqn (101) can be written in the form:

$$\frac{2}{T}\int_0^T \mathbf{K}_t(\mathbf{u})\mathbf{q}(t)\sin z_l\lambda t\,\mathrm{d}t=\mathbf{K}_{il}^{\mathrm{cs}}\mathbf{q}_i^{\mathrm{c}}+\mathbf{K}_{il}^{\mathrm{ss}}\mathbf{q}_i^{\mathrm{s}}. \tag{109}$$

Comparing the right-hand-sides of eqns (101) and (109) we conclude that

$$\mathbf{K}_{li}^{\mathrm{sc}}=\mathbf{K}_{il}^{\mathrm{cs}},\quad \mathbf{K}_{il}^{\mathrm{ss}}=\mathbf{K}_{li}^{\mathrm{ss}}, \tag{110}$$

and moreover, $\tilde{\mathbf{K}}_{il}=\tilde{\mathbf{K}}_{li}$, which yields a symmetry of matrix $\mathbf{K}$.

On a basis of eigenvalues of the quadratic eigenvalue problem (104) the stability of assumed steady-state solution is examined. In the considered case the eigenvalues μ are, in general, the complex numbers. If the real parts of all eigenvalues are negative the considered steady-state solution is stable.

Usually, the stability of a family of steady-state solution depending on some parameter (in many cases it is the frequency of excitation) is examined. On a boundary between stable and unstable solution, in a parametric space, two characteristic cases can occur:

(a) both real and imaginary parts of some eigenvalue are equal to zero, i.e. $\mu = 0$, which means that the saddle-node bifurcation occurs,
(b) only the real parts of two eigenvalues are equal to zero but the imaginary parts are not, i.e. the Hopf bifurcation occurs in this point.

In a first case, on the boundary between the stable and unstable solutions the eigenvalue problem (104) takes a form

$$(\mathbf{K}-\lambda^2\mathbf{X}+\lambda\mathbf{D})\mathbf{q}=\mathbf{0} \tag{111}$$

and the condition of existence of nontrivial solution of (105) is

$$\det(\mathbf{K}-\lambda^2\mathbf{X}+\lambda\mathbf{D})=0. \tag{112}$$

On the basis of the above condition the numerical effort of stability analysis can be significantly reduced. It is assumed that the determinant (112) is a continuous function of excitation frequency λ. If the first steady state solution is determined, its stability is examined on a basis of eigenvalues resulting from the eigenvalue problem (104). Moreover, a sign of determinant (112) can be easily determined. Now it is clear how the sign of the determinant is related to this stable or unstable solution. For the next point on the response curve only the sign of determinant is required in order to examine the stability properties of considered steady-state solution. If the determinant sign is as in the previous state then the solution is stable or unstable as in the previous one. In the opposite case, from continuity of the determinant (112) with respect to λ, it is concluded that the stability properties of the current solution are different in comparison with the previous one. Unfortunately, in this way a point of the Hopf bifurcation cannot be determined.

The proposed stability analysis is very costly from a numerical point of view and any simplification of this analysis is very desirable. Some proposition of simplification is presented below.

Notice, that the motion disturbance $\delta\mathbf{u}(t)$ is described by

$$\delta\mathbf{u}(t) = \mathrm{e}^{\mu t}\mathbf{q}(t) = \mathrm{e}^{\mu t}(\mathbf{q}_i^c \cos z_i\lambda t + \mathbf{q}_i^s \sin z_i\lambda t). \tag{113}$$

where $i = 1, \ldots, m$.

Let us assume, that near the boundary between the stable and unstable solutions the function $e^{\mu t}$ is to be a slowly varying function of time in comparison with $\mathbf{q}(t)$. Now the derivatives of $\delta\mathbf{u}(t)$ could be simplified as follows

$$\delta\dot{\mathbf{u}}(t) = \mu\,\mathrm{e}^{\mu t}\mathbf{q}(t) + \mathrm{e}^{\mu t}\dot{\mathbf{q}}(t) \approx \mathrm{e}^{\mu t}\dot{\mathbf{q}}(t),$$
$$\delta\ddot{\mathbf{u}}(t) = \mu^2\,\mathrm{e}^{\mu t}\mathbf{q}(t) + 2\mu\,\mathrm{e}^{\mu t}\dot{\mathbf{q}}(t) + \mathrm{e}^{\mu t}\ddot{\mathbf{q}}(t) \approx 2\mu\,\mathrm{e}^{\mu t}\dot{\mathbf{q}}(t) + \mathrm{e}^{\mu t}\ddot{\mathbf{q}}(t), \tag{114}$$

and after introducing relations (113) into (87) one obtains

$$\mathbf{M}\ddot{\mathbf{q}}(t) + (2\mu\mathbf{M} + \mathbf{C})\dot{\mathbf{q}}(t) + \mathbf{K}_t(\mathbf{u})\mathbf{q}(t) = \mathbf{0}. \tag{115}$$

The solution of eqn (115) is taken in the form of relation (90) and after application of the Galerkin method the following equations are derived

$$(\mathbf{K}_{li}^{cc} - \alpha_{li} z_i^2 \lambda^2 \mathbf{M})\mathbf{q}_i^c + [\mathbf{K}_{li}^{cs} + z_i\lambda\alpha_{li}(\mathbf{C} + 2\mu\mathbf{M})]\mathbf{q}_i^s = \mathbf{0},$$
$$[\mathbf{K}_{li}^{sc} - z_i\lambda\beta_{li}(\mathbf{C} + 2\mu\mathbf{M})]\mathbf{q}_i^c + (\mathbf{K}_{li}^{ss} - z_i^2\lambda^2\beta_{li}\mathbf{M})\mathbf{q}_i^s = \mathbf{0}, \tag{116}$$

where $i, l = 1, 2, \ldots, m$. The compact form of above equations is

$$[\mathbf{K} - \lambda^2\mathbf{X} + \lambda(\mathbf{D} + 2\mu\mathbf{E})]\mathbf{q} = \mathbf{0}. \tag{117}$$

From a numerical point of view this approach is more attractive because only the linear eigenvalue problem has to be solved. Notice, that the condition (112) also naturally arises from (117) if the saddle-node bifurcation occurs on the boundary between the stable and unstable solutions.

5. DETERMINATION OF THE TANGENT MATRIX $\mathbf{G}_a$

The incremental-iterative procedures are often used to solve the system of nonlinear algebraic equations with parameter like the matrix amplitude eqn (79) appearing in this work. An important part of such techniques is derivation of the incremental form of matrix amplitude equation and the tangent matrix $\mathbf{G}_a$.

Expanding the vector $\mathbf{G}$ given by relation (79) in the Taylor series around some known solution denoted by $\mathbf{a}$ and λ and retaining only the linear terms with respect of increment of $\mathbf{a}$ and λ one obtains

$$\mathbf{G}(\mathbf{a} + \Delta\mathbf{a}, \lambda + \Delta\lambda, \mathbf{P}) \approx \mathbf{G}(\mathbf{a}, \lambda, \mathbf{P}) + \mathbf{G}_a(\mathbf{a}, \lambda, \mathbf{P})\,\Delta\mathbf{a} + \mathbf{G}_\lambda\,\Delta\lambda = \mathbf{0}, \tag{118}$$

where

$$\mathbf{G}_\lambda = \left(\frac{1}{2\lambda}\mathbf{D} - \mathbf{B}\right)\mathbf{a}, \tag{119}$$

and $\mathbf{G}_a$ denotes the vector and matrix of first derivatives of $\mathbf{G}$ with respect to λ^2 and $\mathbf{a}$, respectively.

The tangent matrix $\mathbf{G}_a$ can be derived in two ways. In the first method we directly differentiate the vector $\mathbf{G}$ with respect to $\mathbf{a}$ and λ, whereas the second one avails it of from the integral definitions of vector $\mathbf{G}$ which are rewritten here, for convenience

$$\mathbf{G}_l^c(\mathbf{a},\lambda) \equiv \frac{2}{T}\int_0^T \mathbf{R}(\mathbf{u}(\mathbf{a}),\lambda,t)\cos z_l\lambda t\,\mathrm{d}t,$$

$$\mathbf{G}_l^s(\mathbf{a},\lambda) \equiv \frac{2}{T}\int_0^T \mathbf{R}(\mathbf{u}(\mathbf{a}),\lambda,t)\sin z_l\lambda t\,\mathrm{d}t. \tag{120}$$

The differential of $\mathbf{G}_l^c$ with respect to $\mathbf{a}$ can be written as

$$\mathrm{d}\mathbf{G}_l^c = \frac{2}{T}\int_0^T \left(\frac{\partial \mathbf{R}}{\partial \mathbf{w}}\mathrm{d}\ddot{\mathbf{u}} + \frac{\partial \mathbf{R}}{\partial \mathbf{v}}\mathrm{d}\dot{\mathbf{u}} + \frac{\partial \mathbf{R}}{\partial \mathbf{u}}\mathrm{d}\mathbf{u}\right)\cos z_l\lambda t\,\mathrm{d}t, \tag{121}$$

where $\mathbf{w} = \ddot{\mathbf{u}}$, $\mathbf{v} = \dot{\mathbf{u}}$,

$$\mathrm{d}\ddot{\mathbf{u}} = \frac{\partial \ddot{\mathbf{u}}}{\partial \mathbf{a}}\mathrm{d}\mathbf{a},\quad \mathrm{d}\dot{\mathbf{u}} = \frac{\partial \dot{\mathbf{u}}}{\partial \mathbf{a}}\mathrm{d}\mathbf{a},\quad \mathrm{d}\mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{a}}\mathrm{d}\mathbf{a}. \tag{122}$$

The matrix equation of motion (35) results in

$$\frac{\partial \mathbf{R}}{\partial \mathbf{w}} = \mathbf{M},\quad \frac{\partial \mathbf{R}}{\partial \mathbf{v}} = \mathbf{C}, \frac{\partial \mathbf{R}}{\partial \mathbf{u}} = \frac{\partial \mathbf{F}}{\partial \mathbf{u}} = \mathbf{K}_t(\mathbf{u}) = \mathbf{K}_0 + \mathbf{K}_2(\mathbf{u}) + \mathbf{K}_3(\mathbf{u}) + \mathbf{K}_4(\mathbf{u}) + \mathbf{K}_5(\mathbf{u}), \tag{123}$$

and the brackets of eqn (121) contain a left-hand-side of the incremental equation of motion (36) hence the relation (121) could be rewritten in the form

$$\mathrm{d}\mathbf{G}_l^c = \frac{2}{T}\int_0^T [\mathbf{M}\,\mathrm{d}\ddot{\mathbf{u}} + \mathbf{C}\,\mathrm{d}\dot{\mathbf{u}} + \mathbf{K}_t(\mathbf{u})\,\mathrm{d}\mathbf{u}]\cos z_l\lambda t\,\mathrm{d}t. \tag{124}$$

The perturbation $\mathrm{d}\mathbf{u}$ is described in time by

$$\mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{a}_i\cos z_i\lambda t + \mathrm{d}\mathbf{b}_i\sin z_i\lambda t, \tag{125}$$

where $\mathrm{d}\mathbf{a}_i$ and $\mathrm{d}\mathbf{b}_i$ are the vectors of amplitude increments.

Inserting the assumed solution of perturbation (125) into eqn (124) and integrating its linear part we obtain

$$\frac{2}{T}\int_0^T (\mathbf{M}\,\mathrm{d}\ddot{\mathbf{u}} + \mathbf{D}\,\mathrm{d}\dot{\mathbf{u}})\cos z_l\lambda t\,\mathrm{d}t = \alpha_{il}(-z_i^2\lambda^2\mathbf{M}\,\mathrm{d}\mathbf{a}_i + z_i\lambda\mathbf{C}\,\mathrm{d}\mathbf{b}_i). \tag{126}$$

The remainder terms on the left-hand-side of eqn (124) can be transformed analogously like the nonlinear terms of motion eqn (35). The elements of matrices $\mathbf{K}_3(\mathbf{u})$, $\mathbf{K}_4(\mathbf{u})$, $\mathbf{K}_2(\mathbf{u})$, $\mathbf{K}_5(\mathbf{u})$ are homogeneous and linear or quadratic functions of $\mathbf{u}$, respectively. Moreover, a structure of solution in time domain for $\mathbf{u}$ and $\mathrm{d}\mathbf{u}$ is identical. Using the relations (63) and (65) as the formulas we can write

$$\frac{2}{T}\int_0^T \mathbf{K}_3(\mathbf{u})\,\mathrm{d}\mathbf{u}\cos z_l\lambda t\,\mathrm{d}t = \mathbf{K}_3(\mathbf{a}_i)\,\mathrm{d}\mathbf{a}_j\alpha_{ijl} + \mathbf{K}_3(\mathbf{b}_i)\,\mathrm{d}\mathbf{b}_j\beta_{ijl},$$

$$\frac{2}{T}\int_0^T \mathbf{K}_4(\mathbf{u})\,\mathrm{d}\mathbf{u}\cos z_l\lambda t\,\mathrm{d}t = \mathbf{K}_4(\mathbf{a}_i)\,\mathrm{d}\mathbf{a}_j\alpha_{ijl} + \mathbf{K}_4(\mathbf{b}_i)\,\mathrm{d}\mathbf{b}_j\beta_{ijl},$$

$$\frac{2}{T}\int_0^T \mathbf{K}_2(\mathbf{u})\,\mathrm{d}\mathbf{u}\cos z_l\lambda t\,\mathrm{d}t = \mathbf{K}_2(\mathbf{a}_i,\mathbf{a}_j)\,\mathrm{d}\mathbf{a}_k\alpha_{ijkl}+\mathbf{K}_2(\mathbf{a}_i,\mathbf{b}_j)\,\mathrm{d}\mathbf{b}_k\beta_{ijkl}$$

$$+\mathbf{K}_2(\mathbf{b}_i,\mathbf{a}_j)\,\mathrm{d}\mathbf{b}_k\gamma_{ijkl}+\mathbf{K}_2(\mathbf{b}_i,\mathbf{b}_j)\,\mathrm{d}\mathbf{a}_k\delta_{ijkl},$$

$$\frac{2}{T}\int_0^T \mathbf{K}_5(\mathbf{u})\,\mathrm{d}\mathbf{u}\cos z_l\lambda t\,\mathrm{d}t = \mathbf{K}_5(\mathbf{a}_i,\mathbf{a}_j)\,\mathrm{d}\mathbf{a}_k\alpha_{ijkl}+\mathbf{K}_5(\mathbf{a}_i,\mathbf{b}_j)\,\mathrm{d}\mathbf{b}_k\beta_{ijkl}$$

$$+\mathbf{K}_5(\mathbf{b}_i,\mathbf{a}_j)\,\mathrm{d}\mathbf{b}_k\gamma_{ijkl}+\mathbf{K}_5(\mathbf{b}_i,\mathbf{b}_j)\,\mathrm{d}\mathbf{a}_k\delta_{ijkl}, \tag{127}$$

where the coefficients α_{ijl}, α_{ijkl}, β_{ijl}, β_{ijkl}, γ_{ijl}, γ_{ijkl}, δ_{ijl}, δ_{ijkl} are given by relations (64) and (66), respectively.

Finally, the eqn (124) could be written in the form

$$\mathrm{d}\mathbf{G}_l^{\mathrm{c}} = (\mathbf{K}_{kl}^{\mathrm{cc}} - z_k^2\lambda^2\alpha_{kl}\mathbf{M})\,\mathrm{d}\mathbf{a}_k + (\mathbf{K}_{kl}^{\mathrm{cs}} + z_k\lambda\alpha_{kl}\mathbf{C})\,\mathrm{d}\mathbf{b}_k, \tag{128}$$

where $l = 1,\ldots,n$

$$\mathbf{K}_{kl}^{\mathrm{cc}} = \alpha_{kl}\mathbf{K}_0 + \alpha_{ikl}[\mathbf{K}_3(\mathbf{a}_i)+\mathbf{K}_4(\mathbf{a}_i)] + \alpha_{ijkl}[\mathbf{K}_2(\mathbf{a}_i,\mathbf{a}_j)+\mathbf{K}_5(\mathbf{a}_i,\mathbf{a}_j)]$$
$$+\delta_{ijkl}[\mathbf{K}_2(\mathbf{b}_i,\mathbf{b}_j)+\mathbf{K}_5(\mathbf{b}_i,\mathbf{b}_j)],$$
$$\mathbf{K}_{kl}^{\mathrm{cs}} = \beta_{ikl}[\mathbf{K}_3(\mathbf{b}_i)+\mathbf{K}_4(\mathbf{b}_i)] + \beta_{ijkl}[\mathbf{K}_2(\mathbf{a}_i,\mathbf{b}_j)+\mathbf{K}_5(\mathbf{a}_i,\mathbf{b}_j)]$$
$$+\gamma_{ijkl}[\mathbf{K}_2(\mathbf{b}_i,\mathbf{a}_j)+\mathbf{K}_5(\mathbf{b}_i,\mathbf{a}_j)]. \tag{129}$$

Transforming the second equation of (120) in a similar way we have

$$\mathrm{d}\mathbf{G}_l^{\mathrm{s}} = (\mathbf{K}_{kl}^{\mathrm{sc}} - z_k\lambda\beta_{kl}\mathbf{C})\,\mathrm{d}\mathbf{a}_k + (\mathbf{K}_{kl}^{\mathrm{ss}} - z_k^2\lambda^2\beta_{kl}\mathbf{M})\,\mathrm{d}\mathbf{b}_k, \tag{130}$$

where $l = 1, 2, \ldots, n$

$$\mathbf{K}_{kl}^{\mathrm{sc}} = \delta_{ikl}[\mathbf{K}_3(\mathbf{b}_i)+\mathbf{K}_4(\mathbf{b}_i)] + \mu_{ijkl}[\mathbf{K}_2(\mathbf{a}_i,\mathbf{b}_j)+\mathbf{K}_5(\mathbf{a}_i,\mathbf{b}_j)]$$
$$+\nu_{ijkl}[\mathbf{K}_2(\mathbf{b}_i,\mathbf{a}_j)+\mathbf{K}_5(\mathbf{b}_i,\mathbf{a}_j)],$$
$$\mathbf{K}_{kl}^{\mathrm{ss}} = \gamma_{ikl}[\mathbf{K}_3(\mathbf{a}_i)+\mathbf{K}_4(\mathbf{a}_i)] + \kappa_{ijkl}[\mathbf{K}_2(\mathbf{a}_i,\mathbf{a}_j)+\mathbf{K}_5(\mathbf{a}_i,\mathbf{a}_j)]$$
$$+\omega_{ijkl}[\mathbf{K}_2(\mathbf{b}_i,\mathbf{b}_j)+\mathbf{K}_5(\mathbf{b}_i,\mathbf{b}_j)] + \beta_{kl}\mathbf{K}_0. \tag{131}$$

The matrices $\mathbf{K}_2(\cdot)$, $\mathbf{K}_3(\cdot)$, $\mathbf{K}_4(\cdot)$, $\mathbf{K}_5(\cdot)$, appearing in eqns (129) and (131) are assembled in a well known way from the matrices of finite elements defined by

$$\mathbf{K}_2^{\mathrm{e}}(\mathbf{a}_i^{\mathrm{e}},\mathbf{b}_j^{\mathrm{e}}) = \int_{V_{\mathrm{e}}}\mathbf{B}_1^T(\mathbf{a}_i^{\mathrm{e}})\mathbf{E}\mathbf{B}_1(\mathbf{b}_j^{\mathrm{e}})\,\mathrm{d}V, \quad \mathbf{K}_3^{\mathrm{e}}(\mathbf{a}_i^{\mathrm{e}}) = \int_{V_{\mathrm{e}}}\mathbf{B}_0^T\mathbf{E}\mathbf{B}_1(\mathbf{a}_i^{\mathrm{e}})\,\mathrm{d}V + \int_{V_{\mathrm{e}}}\mathbf{B}_1^T(\mathbf{a}_i^{\mathrm{e}})\mathbf{E}\mathbf{B}_0\,\mathrm{d}V,$$

$$\mathbf{K}_4^{\mathrm{e}}(\mathbf{a}_i^{\mathrm{e}}) = \int_{V_{\mathrm{e}}}\mathbf{G}^{\mathrm{e}T}\mathbf{Z}_l(\mathbf{a}_i^{\mathrm{e}})\mathbf{G}^{\mathrm{e}}\,\mathrm{d}V, \quad \mathbf{K}_5^{\mathrm{e}}(\mathbf{a}_i^{\mathrm{e}},\mathbf{b}_j^{\mathrm{e}}) = \int_{V_{\mathrm{e}}}\mathbf{G}^{\mathrm{e}T}\mathbf{Z}_n(\mathbf{a}_i^{\mathrm{e}},\mathbf{b}_j^{\mathrm{e}})\mathbf{G}^{\mathrm{e}}\,\mathrm{d}V, \tag{132}$$

where the matrix $\mathbf{G}^{\mathrm{e}}$ is identical with one appearing in the relation $\mathbf{B}_1(\mathbf{u}) = \mathbf{A}(\mathbf{u})\mathbf{G}^{\mathrm{e}}$ (see Section 2). The matrices $\mathbf{Z}_l(\mathbf{a}_i^{\mathrm{e}})$ and $\mathbf{Z}_n(\mathbf{a}_i^{\mathrm{e}},\mathbf{b}_j^{\mathrm{e}})$ have a general form as the previously mentioned matrices $\mathbf{Z}_l(\mathbf{u}_{\mathrm{e}})$ and $\mathbf{Z}_n(\mathbf{u}_{\mathrm{e}})$ but now they are built from the vectors $\mathbf{T}_l = \mathbf{T}_l(\mathbf{a}_i^{\mathrm{e}})$ and $\mathbf{T}_n = \mathbf{T}_n(\mathbf{a}_i^{\mathrm{e}},\mathbf{b}_j^{\mathrm{e}})$ which depend on the amplitudes of particular harmonics $\mathbf{a}_i^{\mathrm{e}}$ and $\mathbf{b}_j^{\mathrm{e}}$, respectively.

Taking into account the relations (25)–(31) we can write

$$\mathbf{T}_1(\mathbf{a}_i^e) = \mathbf{E}\mathbf{e}_l(\mathbf{a}_i^e), \quad \mathbf{T}_2(\mathbf{a}_i^e, \mathbf{b}_j^e) = \mathbf{E}\mathbf{e}_n(\mathbf{a}_i^e, \mathbf{b}_j^e), \tag{133}$$

where

$$\mathbf{e}_l(\mathbf{a}_i^e) = \mathbf{B}_0\mathbf{a}_i^e, \quad \mathbf{e}_n(\mathbf{a}_i^e, \mathbf{b}_j^e) = \tfrac{1}{2}\mathbf{B}_1(\mathbf{a}_i)\mathbf{b}_j^e, \quad \mathbf{B}_1(\mathbf{a}_i) = \mathbf{A}(\mathbf{a}_i^e)\mathbf{G}^e.$$

Introducing the following notation

$$\mathrm{d}\tilde{\mathbf{G}}_l = \mathrm{col}(\mathrm{d}\mathbf{G}_l^c, \mathrm{d}\mathbf{G}_l^s), \quad \mathrm{d}\tilde{\mathbf{a}}_k = \mathrm{col}(\mathrm{d}\mathbf{a}_k, \mathrm{d}\mathbf{b}_k),$$

$$\tilde{\mathbf{D}}_{kl} = \begin{bmatrix} \mathbf{0} & \alpha_{kl}\mathbf{C} \\ -\beta_{kl}\mathbf{C} & \mathbf{0} \end{bmatrix}, \quad \tilde{\mathbf{K}}_{kl}(\mathbf{a}) = \begin{bmatrix} \mathbf{K}_{kl}^{cc} & \mathbf{K}_{kl}^{cs} \\ \mathbf{K}_{kl}^{sc} & \mathbf{K}_{kl}^{ss} \end{bmatrix}, \quad \tilde{\mathbf{B}}_{kl} = \begin{bmatrix} \alpha_{kl}\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \beta_{kl}\mathbf{M} \end{bmatrix}, \tag{134}$$

and subsequently

$$\mathrm{d}\mathbf{G} = \mathrm{col}(\mathrm{d}\tilde{\mathbf{G}}_1, \ldots, \mathrm{d}\tilde{\mathbf{G}}_n), \quad \mathrm{d}\mathbf{a} = \mathrm{col}(\mathrm{d}\tilde{\mathbf{a}}_1, \ldots, \mathrm{d}\tilde{\mathbf{a}}_n),$$

$$\mathbf{D} = \lceil z_1\tilde{\mathbf{D}}_{11}, \ldots, z_n\tilde{\mathbf{D}}_{nn} \rfloor, \quad \mathbf{B} = \lceil z_1^2\tilde{\mathbf{B}}_{11}, \ldots, z_n^2\tilde{\mathbf{B}}_{nn} \rfloor, \quad \mathbf{K}(\mathbf{a}) = \begin{bmatrix} \tilde{\mathbf{K}}_{11}(\mathbf{a}), \ldots, \tilde{\mathbf{K}}_{1n}(\mathbf{a}) \\ \vdots \qquad \vdots \\ \tilde{\mathbf{K}}_{n1}(\mathbf{a}), \ldots, \tilde{\mathbf{K}}_{nn}(\mathbf{a}) \end{bmatrix}, \tag{135}$$

the eqns (128) and (130) can be written in the form

$$\mathrm{d}\mathbf{G}(\mathbf{a}) = [\mathbf{K}(\mathbf{a}) + \lambda\mathbf{D} - \lambda^2\mathbf{B}]\,\mathrm{d}\mathbf{a}, \tag{136}$$

which means that the tangent matrix $\mathbf{G}_a$ is given by

$$\mathbf{G}_a = \mathbf{K}(\mathbf{a}) + \lambda\mathbf{D} - \lambda^2\mathbf{B}. \tag{137}$$

In the case of undamped free vibration

$$\mathrm{d}\mathbf{G}(\mathbf{a}) = [\mathbf{K}(\mathbf{a}) - \lambda^2\mathbf{B}]\,\mathrm{d}\mathbf{a} = \mathbf{G}_a\,\mathrm{d}\mathbf{a}, \tag{138}$$

where $\mathrm{d}\mathbf{G}$, $\mathbf{K}(\mathbf{a})$, $\mathbf{B}$, $\mathrm{d}\mathbf{a}$ are given by relation (135) and now

$$\mathrm{d}\tilde{\mathbf{G}}_l = \mathrm{d}\mathbf{G}_l^c, \quad \mathrm{d}\tilde{\mathbf{a}}_k = \mathrm{d}\mathbf{a}_k, \quad \tilde{\mathbf{K}}_{kl}(\mathbf{a}) = \mathbf{K}_{kl}^{cc}, \quad \tilde{\mathbf{B}}_{kl} = \alpha_{kl}\mathbf{M}, \quad \mathbf{K}_{kl}^{cc} = \alpha_{ikl}[\mathbf{K}_3(\mathbf{a}_i) + \mathbf{K}_4(\mathbf{a}_i)]$$
$$+ \alpha_{ijkl}[\mathbf{K}_2(\mathbf{a}_i, \mathbf{a}_j) + \mathbf{K}_5(\mathbf{a}_i, \mathbf{a}_j)] + \alpha_{kl}\mathbf{K}_0. \tag{139}$$

The procedure of calculation of the matrix $\mathbf{K}(\mathbf{a})$ consists of a few steps which are summarized as a pseudo code below:

Step 1. Calculate or read the matrix $\mathbf{K}_0$,

for l from 1 to n,
set $\mathbf{K}_{lk}^{cc} := \mathbf{K}_{lk}^{ss} := \mathbf{K}_{lk}^{sc} := \mathbf{K}_{lk}^{cs} := \mathbf{0}$,
for k from 1 to n.

Step 2. If k is equal to l then set $\mathbf{K}_{lk}^{cc} := \alpha_{lk}\mathbf{K}_0$, $\mathbf{K}_{lk}^{ss} := \beta_{lk}\mathbf{K}_0$,

for j from 1 to n,
for i from 1 to n.

Step 3. Calculate the matrices

$$\mathbf{K}_3(\mathbf{a}_i), \mathbf{K}_3(\mathbf{b}_i), \mathbf{K}_4(\mathbf{a}_i), \mathbf{K}_4(\mathbf{b}_i), \mathbf{K}_2(\mathbf{a}_i, \mathbf{a}_j), \mathbf{K}_2(\mathbf{b}_i, \mathbf{b}_j), \mathbf{K}_2(\mathbf{a}_i, \mathbf{b}_j), \mathbf{K}_2(\mathbf{b}_i, \mathbf{a}_j),$$
$$\mathbf{K}_5(\mathbf{a}_i, \mathbf{a}_j), \mathbf{K}_5(\mathbf{b}_i, \mathbf{b}_j), \mathbf{K}_5(\mathbf{a}_i, \mathbf{b}_j), \mathbf{K}_5(\mathbf{b}_i, \mathbf{a}_j),$$

and after it set

$$\bar{\mathbf{K}}_{kl}^{cc} := \alpha_{ikl}[\mathbf{K}_3(\mathbf{a}_i)+\mathbf{K}_4(\mathbf{a}_i)]+\alpha_{ijkl}[\mathbf{K}_2(\mathbf{a}_i, \mathbf{a}_j)+\mathbf{K}_5(\mathbf{a}_i, \mathbf{a}_j)]$$
$$+\delta_{ijkl}[\mathbf{K}_2(\mathbf{b}_i, \mathbf{b}_j)+\mathbf{K}_5(\mathbf{b}_i, \mathbf{b}_j)],$$
$$\bar{\mathbf{K}}_{kl}^{cs} := \beta_{ikl}[\mathbf{K}_3(\mathbf{b}_i)+\mathbf{K}_4(\mathbf{b}_i)]+\beta_{ijkl}[\mathbf{K}_2(\mathbf{a}_i, \mathbf{b}_j)+\mathbf{K}_5(\mathbf{a}_i, \mathbf{b}_j)]$$
$$+\gamma_{ijkl}[\mathbf{K}_2(\mathbf{b}_i, \mathbf{a}_j)+\mathbf{K}_5(\mathbf{b}_i, \mathbf{a}_j)].$$
$$\bar{\mathbf{K}}_{kl}^{sc} := \delta_{ikl}[\mathbf{K}_3(\mathbf{b}_i)+\mathbf{K}_4(\mathbf{b}_i)]+\mu_{ijkl}[\mathbf{K}_2(\mathbf{a}_i, \mathbf{b}_j)+\mathbf{K}_5(\mathbf{a}_i, \mathbf{b}_j)]$$
$$+\nu_{ijkl}[\mathbf{K}_2(\mathbf{b}_i, \mathbf{a}_j)+\mathbf{K}_5(\mathbf{b}_i, \mathbf{a}_j)],$$
$$\bar{\mathbf{K}}_{kl}^{ss} := \gamma_{ikl}[\mathbf{K}_3(\mathbf{a}_i)+\mathbf{K}_4(\mathbf{a}_i)]+\kappa_{ijkl}[\mathbf{K}_2(\mathbf{a}_i, \mathbf{a}_j)+\mathbf{K}_5(\mathbf{a}_i, \mathbf{a}_j)]$$
$$+\omega_{ijkl}[\mathbf{K}_2(\mathbf{b}_i, \mathbf{b}_j)+\mathbf{K}_5(\mathbf{b}_i, \mathbf{b}_j)].$$

Step 4. Set $\mathbf{K}_{kl}^{cc} := \mathbf{K}_{kl}^{cc}+\bar{\mathbf{K}}_{kl}^{cc}(\mathbf{a}_i, \mathbf{b}_j)$, $\mathbf{K}_{kl}^{cs} := \mathbf{K}_{kl}^{cs}+\bar{\mathbf{K}}_{kl}^{cs}(\mathbf{a}_i, \mathbf{b}_j)$, $\mathbf{K}_{kl}^{sc} := \mathbf{K}_{kl}^{sc}+\bar{\mathbf{K}}_{kl}^{sc}(\mathbf{a}_i, \mathbf{b}_j)$, $\mathbf{K}_{kl}^{ss} := \mathbf{K}_{kl}^{ss}+\bar{\mathbf{K}}_{kl}^{ss}(\mathbf{a}_i, \mathbf{b}_j)$,

end for i,
end for j,
end for k,
end for l.

If the assumed form of solution of the motion equation contains the constant term and for example $z_n = 0$ we must change the definitions of several vectors and matrices. Now $\tilde{\mathbf{a}}_n = \mathbf{a}_n$, $\mathrm{d}\tilde{\mathbf{a}}_n = \mathrm{d}\mathbf{a}_n$,

$$\tilde{\mathbf{K}}_{nl}(\mathbf{a}) = \begin{bmatrix}\mathbf{K}_{nl}^{cc}\\ \mathbf{K}_{nl}^{sc}\end{bmatrix}, \quad \tilde{\mathbf{K}}_{ln}(\mathbf{a}) = [\mathbf{K}_{ln}^{cc}, \mathbf{K}_{ln}^{cs}], \quad \tilde{\mathbf{D}}_{nn} = \mathbf{0}, \quad \tilde{\mathbf{B}}_{nn} = \alpha_{nn}\mathbf{M}.$$

Moreover, during the process of calculation of $\mathbf{K}_{nl}^{cc}$, $\mathbf{K}_{nl}^{sc}$ and $\mathbf{K}_{ln}^{cs}$ we must formally insert $\mathbf{b}_n = \mathbf{0}$.

The matrix $\mathbf{G}_a$ contains several strips in which the nonzero elements are located as shown by the lined areas in Fig. 1. This fact offers a possibility of significant reduction of computer memory required. Moreover, the parallel computers can be used in a natural way to calculate the matrix $\mathbf{G}_a$ and to solve the incremental amplitude eqn (118).

6. CONCLUDING REMARKS

In this paper, the theoretical background of computational method for free and steady state vibration of geometrically non-linear structures is presented. The Galerkin method assures that the systems with strong non-linearity could be solved with an appropriate accuracy. In the proposed method many harmonics are taken into account in the solution of motion equation which means that the main and secondary resonances can be analyzed in a uniform way. The resulting matrix amplitude equation and the corresponding tangent matrix are derived and given in the explicit forms. Moreover, the stability of the steady state solution is considered and some possible simplifications of stability analysis are suggested. All of these make the proposed method general, complete and attractive from a computational point of view. In a companion paper by Lewandowski (1996) several numerical results show that the method is effective, efficient and accurate.

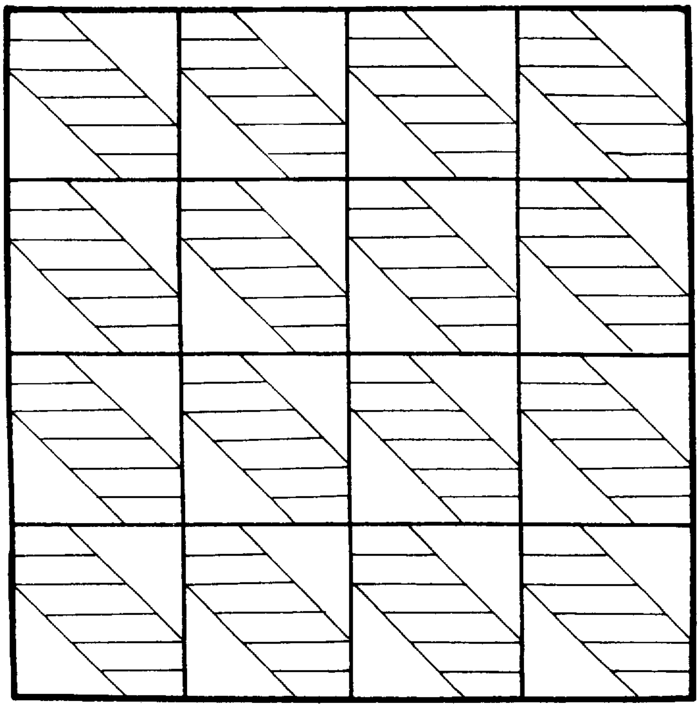

Fig. 1. A structure of matrix $\mathbf{G}_a$.

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